Math 233, Exam 2, October 19th

This exam should have 16 questions. All questions carry equal marks.

Mark your ID number on the six blank lines on the top of your answer card, using one line for each digit. Print your name on the top of the card.

Choose the answer that is closest to the solution. Mark your answer card with a PENCIL by shading in the correct box.

You may use a calculator, but not one that has a graphing function. Make sure that angles are set to radians.

1. Let

\[
A = \begin{pmatrix}
1 & 1.1 & 1.2 \\
1.3 & 1.4 & 1.5 \\
1.6 & 1.7 & 1.8
\end{pmatrix}, \quad B = \begin{pmatrix}
2 & 2.1 & 2.2 \\
2.3 & 2.4 & 2.5 \\
2.6 & 2.7 & 2.8
\end{pmatrix}.
\]

What is the (2, 3) entry of \(AB\)?

A) 10.0
B) 10.1
C) 10.2
D) 10.3
E) 10.4
F) 10.5
G) 10.6
H) 10.7
I) 10.8
J) 10.9

\[
1.3 \times 2.2 + 1.4 \times 2.5 + 1.5 \times 2.8 = 10.56
\]

2. Find the unit tangent vector \(T\) to the curve \(\mathbf{r}(t) = (-2 \cos(t), 3 \sin(t), e^t)\) at \(t = 1\). What is the first component of \(T\)?

\[
\mathbf{r}'(t) = \left( 2 \sin t, 3 \cos t, e^t \right)
\]

\[
T(t) = \frac{\mathbf{r}'(t)}{||\mathbf{r}'(t)||} = \frac{\left( 2 \sin 1, 3 \cos 1, e \right)}{\sqrt{4 \sin^2 1 + 9 \cos^2 1 + e^2}}
\]

\[
\frac{2 \sin 1}{\sqrt{4 \sin^2 1 + 9 \cos^2 1 + e^2}} \approx .47
\]

1
3. What is the length of the acceleration vector of a particle whose position is given by \( \mathbf{r}(t) = (3t^2, 3\sin(t), e^t) \) at time \( t = 1 \)?

A) 7.0  \hspace{1cm} B) 7.1
C) 7.2  \hspace{1cm} D) 7.3
E) 7.4  \hspace{1cm} F) 7.5
G) 7.6  \hspace{1cm} H) 7.7
I) 7.8  \hspace{1cm} J) 7.9

\[ \mathbf{r}'(t) = (6t, 3\cos(t), e^t) \]

\[ \mathbf{r}''(t) = (6, -3\sin(t), e^t) \]

\[ \mathbf{r}''(1) = (6, -3\sin 1, e) \]

\[ |\mathbf{r}''(1)| = \sqrt{36 + 9\sin^2 1 + e^2} \approx 7.1 \]

4. Let \( f(x, y) = y^5 - 3xy \). Find \( \frac{\partial f}{\partial y} \) at \( (2.1, 3.1) \).

A) 455.0  \hspace{1cm} B) 455.1
C) 455.2  \hspace{1cm} D) 455.3
E) 455.4  \hspace{1cm} F) 455.5
G) 455.6  \hspace{1cm} H) 455.7
I) 455.8  \hspace{1cm} J) 455.9

\[ \frac{\partial f}{\partial y} = 5y^4 - 3x \]

\[ \frac{\partial f}{\partial y}(2.1, 3.1) = 5(3.1)^4 - 3(2.1) = 455.5 \]
5. Let \( f(x, y) = e^{x^2 \sin(y)} \). Find \( \frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x} \) at the point \((1, \pi/4)\).

\[ \text{Clairaut's theorem } \Rightarrow \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} \]

6. If

\[ x^2 + y^2 + z^2 = 3xyz - 19, \]

what is \( \frac{\partial z}{\partial x} \) at \((2, 2, 3)\)?
7. Find the tangent plane approximation to the function \( f(x,y) = \frac{x}{x+2y} \) at the point \((1,2)\). What is this approximation at \((1.5,2.5)\)?

A) 0.2
B) 0.21
C) 0.22
D) 0.23
E) 0.24

\[ L(x,y) = f(1,2) + \nabla f(1,2) \left( \frac{x-1}{y-2} \right) \]

where

\[ \nabla f(1,2) = \begin{pmatrix} f_x(1,2) \\ f_y(1,2) \end{pmatrix}, \]

\[ f_x = \frac{x+2y-x}{(x+2y)^2} = \frac{2y}{(x+2y)^2}, \quad f_y = \frac{-2x}{(x+2y)^2}. \]

J) 0.28

\[ L(1.5,2.5) = \frac{1}{5} + \begin{pmatrix} 4/25 \\ -2/25 \end{pmatrix} \begin{pmatrix} 1/5 \end{pmatrix} = 0.24 \]

8. Let \( g(u,v) \) be a function from \( \mathbb{R}^2 \) to \( \mathbb{R}^2 \), and \( f(x,y) \) a function from \( \mathbb{R}^2 \) to \( \mathbb{R} \). Suppose

\[ g(0,1) = (2,1), \quad Dg(0,1) = \begin{pmatrix} 2 & 3 \\ 1 & -2 \end{pmatrix} \]

and

\[ \frac{\partial f}{\partial x}(0,1) = 2, \quad \frac{\partial f}{\partial y}(0,1) = 3, \quad \frac{\partial f}{\partial x}(2,1) = 4, \quad \frac{\partial f}{\partial y}(2,1) = -1. \]

Then the derivative of \( f \circ g \) at \((0,1)\) is:

A) (4,-1)
B) (4,0)
C) (4,4)
D) (4,6)
E) (6,0)
F) (6,7)
G) (6,14)
H) (7,0)
I) (7,7)
J) (7,14)

\[ D(f \circ g)(0,1) \frac{\partial g}{\partial x}(0,1) = Df(g(0,1)) \cdot Dg(0,1) \]

\[ = \begin{pmatrix} 2 & 3 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} 2 & 3 \\ 1 & -2 \end{pmatrix} \]

\[ = \begin{pmatrix} 4 & -1 \\ 1 & -2 \end{pmatrix} \]

\[ = (7,14) \]
9. Let \( \mathbf{r}(t) = (2t, t^2, \frac{1}{3}t^3) \), \( 0 \leq t \leq 1 \). What is the arc length of the curve?

\[
\mathbf{r}'(t) = (2, 2t, t^2)
\]

\[
|\mathbf{r}'(t)| = \sqrt{4 + 4t^2 + t^4} = t^2 + 2
\]

\[
\int_0^1 |\mathbf{r}'(t)| \, dt = \int_0^1 (t^2 + 2) \, dt = \left[ \frac{1}{3}t^3 + 2t \right]_0^1 = 2.33
\]

10. Let \( f(x, y, z) = xe^{2yz} \). Find the rate of change of \( f \) at \( (3, 2, 0) \) in the direction \( \left( \frac{2}{3}, -\frac{2}{3}, \frac{1}{3} \right) \).

\[
\nabla f = (e^{2yz}, 2z \cdot e^{2yz}, 2y \cdot e^{2yz})
\]

\[
\nabla f(3, 2, 0) = \left( 1, 0, 12 \right)
\]

\[
\nabla f(3, 2, 0) \cdot \left( \frac{2}{3}, -\frac{2}{3}, \frac{1}{3} \right) = 4.66
\]
11. How many critical points does the function \( f(x, y) = y^3 - 2y \cos x \) have in the rectangle \([0, 4\pi] \times [0, 4\pi]\)?

A) 0
B) 1
C) 2
D) 3
E) 4
F) 5

\[ f_x = 2y \sin x \]
\[ f_y = 3y^2 - 2 \cos x \]

\[ f_x = 0 \] when \( y = 0 \) or \( x = 0, \pi, 2\pi, 3\pi, 4\pi \)

If \( y = 0 \), then for \( f_y = 0 \), \( x \) must be \( \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \) or \( \frac{7\pi}{2} \)

If \( x = 0 \) then for \( f_y = 0 \), \( y \) must be \( \sqrt{\frac{2}{3}} \)

If \( x = \pi \) then \( f_y > 0 \) for all \( y \). Similarly for \( x = 3\pi \).

\[ \therefore \text{critical points are } \left( \frac{\pi}{2}, 0 \right), \left( \frac{3\pi}{2}, 0 \right), \left( \frac{5\pi}{2}, 0 \right), \left( \frac{7\pi}{2}, 0 \right), \left( 0, \sqrt{\frac{2}{3}} \right), \left( 2\pi, \sqrt{\frac{2}{3}} \right), \left( 4\pi, \sqrt{\frac{2}{3}} \right) \]

12. The function \( f(x, y) = x^6 + y^6 - 6xy \) has three critical points. They are:

A) 3 local maxima
B) 2 local maxima, 1 saddle point
C) 2 local maxima, 1 local minimum
D) 1 local maximum, 2 saddle points
E) 1 local maximum, 1 saddle point, 1 local minimum
F) 1 local maximum, 2 local minima
G) 3 saddle points
H) 2 saddle points, 1 local minimum
I) 1 saddle point, 2 local minima
J) 3 local minima

\[ f_x = 6x^5 - 6y \]
\[ f_y = 6y^5 - 6x \]
\[ f_{xx} = 30x^4 \]
\[ f_{yy} = 30y^4 \]
\[ f_{xy} = -6 \]

critical points are \( (0,0), (1,1), (-1,-1) \)

\[ D(x, y) = 900x^4y^4 - 36 \]

\[ D(0,0) < 0 \Rightarrow (0,0) \text{ is saddle point} \]

\[ D(1,1) > 0 \text{ and } f_{xx}(1,1) > 0 \Rightarrow (1,1) \text{ local min} \]

\[ D(-1,-1) > 0 \text{ and } f_{xx}(-1,-1) > 0 \Rightarrow (-1,-1) \text{ local min} \]
13. Consider the following 3 statements:

(I) If \( \frac{df}{dx} \) and \( \frac{df}{dy} \) are continuous, then \( f \) is differentiable.  \( \text{True} \)

(II) If \( f(x, y) \) is continuous as a function of \( x \) for each fixed \( y \), and continuous as a function of \( y \) for each fixed \( x \), then \( f \) is continuous.  \( \text{False} \)

(III) If \( g : \mathbb{R}^n \rightarrow \mathbb{R}^m \) is differentiable at a point \( a \), and \( f : \mathbb{R}^n \rightarrow \mathbb{R}^p \) is differentiable at \( g(a) \), then \( f \circ g \) is differentiable at \( a \).  \( \text{True if } m = n \)

Then:

A. All three statements are false.
B. Statement (I) is true, the other two are false.
C. Statement (II) is true, the other two are false.
D. Statement (III) is true, the other two are false.
E. Statement (I) is false, the other two are true.
F. Statement (II) is false, the other two are true.
G. Statement (III) is false, the other two are true.
H. All three statements are true.

\[
\begin{align*}
f(x, y) &= \begin{cases} 
\frac{xy}{x^2 + y^2}, & (x, y) \neq (0, 0) \\
0, & (x, y) = (0, 0)
\end{cases} \\
\text{can show } f(x, y) \rightarrow \frac{1}{2} a e \\
(x, y) \rightarrow (0, 0) \text{ along } y = x
\end{align*}
\]

14. Consider the following 3 statements:

(I) Matrix multiplication is associative.  \( \text{True} \)

(II) Matrix multiplication is commutative.  \( \text{False} \)

(III) Matrix addition is commutative.  \( \text{True} \)

Then:

A. All three statements are false.
B. Statement (I) is true, the other two are false.
C. Statement (II) is true, the other two are false.
D. Statement (III) is true, the other two are false.
E. Statement (I) is false, the other two are true.
F. Statement (II) is false, the other two are true.
G. Statement (III) is false, the other two are true.
H. All three statements are true.
15. Let \( f(x, y, z) \) be a differentiable function. The direction of most rapid decrease of \( f \) at a point \( a \) is:

A. In the direction \( \nabla f(a) \).
B. Perpendicular to \( \nabla f(a) \).
C. In the direction \( -\nabla f(a) \).
D. In the direction \( a \).
E. In the direction \( -a \).
F. In the direction \( \nabla f(a) \times a \).
G. In the direction \( -\nabla f(a) \times a \).

\[
D_u f = \nabla f \cdot u = |\nabla f| \cos \theta
\]

is maximized when \( \cos \theta = 1 \), i.e. \( \theta = 0 \),
i.e. \( u \) points in direction of \( -\nabla f \).

16. Let \( F : \mathbb{R}^2 \to \mathbb{R}^3 \). Let \( a \) be a point in \( \mathbb{R}^2 \). Then the derivative of \( F \) at \( a \) is:

A) A number
B) A vector in \( \mathbb{R}^2 \)
C) A vector in \( \mathbb{R}^3 \)
D) A 2-by-2 matrix.
E) A 2-by-3 matrix.
F) A 3-by-2 matrix.
G) A 3-by-3 matrix.
H) A 5-by-5 matrix
I) A 6-by-6 matrix.

\[
dF(a) \text{ must map a column vector with 2 components to a column vector with 3 components}
\]