Math 233, Exam 3, November 16th

This exam should have 16 questions. All questions carry equal marks.
Mark your ID number on the six blank lines on the top of your answer card, using one line for each digit. Print your name on the top of the card.
Choose the answer that is closest to the solution. Mark your answer card with a PENCIL by shading in the correct box.
You may use a calculator, but not one that has a graphing function. Make sure that angles are set to radians.

1. Find the average value of $e^{x+2y}$ on the rectangle $[0, 2] \times [0, 1]$.

A) 10.0
B) 10.1
C) 10.2
D) 10.3
E) 10.4
F) 10.5
G) 10.6
H) 10.7
I) 10.8
J) 10.9

$\frac{1}{2} \int_0^1 \int_0^2 e^{x+2y} \, dx \, dy$

$= \frac{1}{2} \left( \int_0^1 e^{2y} \, dy \right) \left( \int_0^2 e^x \, dx \right)$

$= \frac{1}{2} \left( \frac{1}{2} (e^2 - 1) \right) \left( e^2 - 1 \right)$

$= 10.2$

2. Calculate $\int_0^2 \int_0^3 (1 + x \sqrt{y}) \, dx \, dy$.

A) 14.40
B) 14.41
C) 14.42
D) 14.43
E) 14.44
F) 14.45
G) 14.46
H) 14.47
I) 14.48
J) 14.49

$\int_0^2 \int_0^3 \left(1 + x \sqrt{y}\right) \, dx \, dy$

$= \int_0^2 \left[ x + \frac{1}{2} x^2 \sqrt{y} \right]_0^3 \, dy$

$= \int_0^2 \left[ 3 + \frac{9}{2} \sqrt{y} \right] \, dy$

$= \left[ 3y + 3y^{3/2} \right]_0^2$

$= 3(2 + 2^{3/2})$

$= 14.49$
3. Let \( g(x, y, z) = x^2 + 2y^4 + 3z^6 \); let \( K = \{(x, y, z) : g(x, y, z) \leq 4\} \), and let \( S = \{(x, y, z) : g(x, y, z) = 4\} \). Let \( f(x, y, z) \) be a differentiable function on \( \mathbb{R}^3 \).

Consider the following 3 statements:

(I) The maximum value of \( f \) on \( K \) always occurs at a unique point.  \( \text{f could be constant} \)

(II) The maximum value of \( f \) on \( K \) always occurs on \( S \).  \( \text{could occur on interior of } K \)

(III) The maximum value of \( f \) on \( K \) always occurs at a point where the gradient of \( f \) is a non-zero multiple of the gradient of \( g \).  \( \text{could occur at a local max. in which case } \nabla f = 0 \).

Then:

\[ \circ \text{A} \text{ All three statements are false.} \]
\[ \text{B. Statement (I) is true, the other two are false.} \]
\[ \text{C. Statement (II) is true, the other two are false.} \]
\[ \text{D. Statement (III) is true, the other two are false.} \]
\[ \text{E. Statement (I) is false, the other two are true.} \]
\[ \text{F. Statement (II) is false, the other two are true.} \]
\[ \text{G. Statement (III) is false, the other two are true.} \]
\[ \text{H. All three statements are true.} \]

4. With the same notation as Problem 3, consider the following 3 statements:

(I) The maximum value of \( f \) on \( S \) always occurs at a unique point.  \( \text{f could be constant} \)

(II) The maximum value of \( f \) on \( S \) always occurs at a point where the gradient of \( f \) is a non-zero multiple of the gradient of \( g \).  \( \text{could occur at a local max.} \)

(III) The maximum value of \( f \) on \( S \) always occurs at a point where the gradient of \( f \) is a multiple of the gradient of \( g \).

Then:

\[ \circ \text{A. All three statements are false.} \]
\[ \text{B. Statement (I) is true, the other two are false.} \]
\[ \text{C. Statement (II) is true, the other two are false.} \]
\[ \text{D. Statement (III) is true, the other two are false.} \]
\[ \text{E. Statement (I) is false, the other two are true.} \]
\[ \text{F. Statement (II) is false, the other two are true.} \]
\[ \text{G. Statement (III) is false, the other two are true.} \]
\[ \text{H. All three statements are true.} \]
5. Find the maximum value of the function \(xyz\) on the set
\[x^2 + 2y^2 + 4z^2 = 9.\]

A) 1.80  
B) 1.81  
C) 1.82  
D) 1.83  
E) 1.84  
F) 1.85  
G) 1.86  
H) 1.87  
I) 1.88  
J) 1.89

If \(x=0\) then \(y,z=0\), which contradicts constraint.
Similarly if \(y\) or \(z=0\). Assume, \(x,y,z \neq 0\). Then
\[
\frac{y^2}{2x} = \frac{x^2}{4y} = \frac{xy}{8z}
\]
\[
y^2 = 2y^2, 2z^2 = y^2
\]
\[
\Rightarrow 6y^2 = 9 \Rightarrow y = \pm \sqrt{\frac{3}{2}} \Rightarrow x = \pm \sqrt{3}, z = \pm \frac{\sqrt{6}}{2} \Rightarrow \max \ of \ xyz
\]
\[
i \in (\frac{\sqrt{3}}{2})(\sqrt{3})(\frac{\sqrt{6}}{2}) = 1.84
\]

6. Evaluate the double integral \(\int_D xy \, dy \, dx\), where \(D\) is the triangular region with vertices \((0,0)\), \((5,0)\) and \((0,5)\).

A) 26.01  
B) 26.02  
C) 26.03  
D) 26.04

\[
\int_0^5 \int_0^{5-x} xy \, dy \, dx
\]
\[
= \int_0^5 \frac{1}{2}xy^2 \bigg|_0^{5-x} \, dx
\]
\[
= \frac{1}{2} \int_0^5 x(5-x)^2 \, dx
\]
\[
= \frac{1}{2} \int_0^5 (25x - 10x^2 + x^3) \, dx
\]
\[
= \frac{1}{2} \left( \frac{25}{2}x^2 - \frac{10}{3}x^3 + \frac{1}{4}x^4 \right) \bigg|_0^5
\]
\[
= 26.04
\]
7. Calculate the volume of the region above the $xy$-plane and below the graph $z = 5 - x^2 - y^2$.

\[
\int_0^{2\pi} \int_0^{\sqrt{5}} (5 - r^2) r \, dr \, d\theta
\]

\[
= \int_0^{2\pi} \left( \frac{5}{2} r^2 - \frac{1}{4} r^4 \right) \bigg|_0^{\sqrt{5}} \, d\theta
\]

\[
= 2\pi \left( \frac{5\sqrt{5}}{2} - \frac{25}{4} \right)
\]

\[
= 39.27
\]

8. Evaluate the integral

\[
\int \int_D (x^2 + y^2)^{3/2} \, dA
\]

where $D$ is the disk $x^2 + y^2 \leq 4$.

\[
= \int_0^{2\pi} \int_0^{2} r^3 \, r \, dr \, d\theta
\]

\[
= \frac{1}{4} \int_0^{2\pi} r^5 \bigg|_0^{2} \, d\theta
\]

\[
= \frac{64\pi}{5}
\]

\[
= 40.21
\]
9. Suppose the joint density function of the random variables $X$ and $Y$ is given by

$$f(x, y) = x^2 + \frac{8}{3} xy$$

if both $x$ and $y$ are between 0 and 1, and $f$ is 0 otherwise. What is the probability that $X \geq 0.6$?

$$1 - P(X < 0.6)$$

$$= 1 - \int_0^{0.6} \int_0^1 (x^2 + \frac{8}{3} xy) \, dy \, dx$$

$$= 1 - \int_0^{0.6} \left( x^2 + \frac{4}{3} x \right) \, dx$$

$$= 1 - \left( \frac{1}{3} (0.6)^3 + \frac{2}{3} (0.6)^2 \right)$$

A) 0.60  
B) 0.61  
C) 0.62  
D) 0.63  
E) 0.64  
F) 0.65  
G) 0.66  
H) 0.67  
I) 0.68  
J) 0.69

10. Find the area of the part of the plane $x + 2y + 2z = 1$ that lies inside the cylinder $x^2 + y^2 = 3$.

$$\int_0^{2\pi} \int_0^{\sqrt{3}} \sqrt{1 + \left( -\frac{1}{2} \right)^2 + (-1)^2} \, r \, dr \, d\theta$$

$$= \frac{1.5}{2} \int_0^{2\pi} r^2 \bigg|_0^{\sqrt{3}} \, d\theta$$

$$= \frac{1.5}{2} \times 3 \times 2\pi$$

= 14.14
11. Find the area of the surface of the paraboloid \( z = x^2 + y^2 \) that lies above the unit disk centered at the origin.

\[
\begin{align*}
\text{A)} & \quad 5.30 \\
\text{B)} & \quad 5.31 \\
\text{C)} & \quad 5.32 \\
\text{D)} & \quad 5.33 \\
\text{E)} & \quad 5.34 \\
\text{F)} & \quad 5.35 \\
\text{G)} & \quad 5.36 \\
\text{H)} & \quad 5.37 \\
\text{J)} & \quad 5.39
\end{align*}
\]

\[
\begin{align*}
= & \int_{0}^{2\pi} \int_{0}^{1} \sqrt{1 + (2r\cos\theta)^2 + (2r\sin\theta)^2} \quad r \, dr \, d\theta \\
= & \int_{0}^{2\pi} \int_{0}^{1} \sqrt{1 + 4r^2} \quad r \, dr \, d\theta \\
= & \frac{1}{8} \int_{0}^{2\pi} \int_{1}^{5} \sqrt{u} \quad dud\theta \\
= & \frac{1}{8} \times 2\pi \times \frac{2}{3} \left( 5^{3/2} - 1 \right) = 5.33
\end{align*}
\]

12. Evaluate the triple integral \( \iiint_{E} (2x + 2y - 4z) \, dV \) over the cube \( E = [0, 1] \times [0, 1] \times [0, 1] \).

\[
\begin{align*}
\text{A)} & \quad -4 \\
\text{B)} & \quad -3 \\
\text{C)} & \quad -2 \\
\text{D)} & \quad -1 \\
\text{E)} & \quad 0 \\
\text{F)} & \quad 1 \\
\text{G)} & \quad 2 \\
\text{H)} & \quad 3 \\
\text{I)} & \quad 4 \\
\text{J)} & \quad 5
\end{align*}
\]

\[
\begin{align*}
\text{Note:} \quad \iiint_{E} x \, dV &= \iiint_{E} y \, dV = \iiint_{E} z \, dV \\
\Rightarrow \quad \iiint_{E} (2x + 2y - 4z) \, dV &= 2 \iiint_{E} x \, dV + 2 \iiint_{E} y \, dV \\
& \quad - 4 \iiint_{E} z \, dV = 0
\end{align*}
\]
13. Evaluate the triple integral \( \iiint_E e^{x+y} \sin(x) \, dV \) over the cube \( E = [-1,1] \times [-1,1] \times [-1,1] \).

\[
\begin{align*}
\mathcal{O} & \quad 0.0 \\
B) & \quad 0.1 \\
C) & \quad 0.2 \\
D) & \quad 0.3 \\
E) & \quad 0.4 \\
F) & \quad 0.5 \\
G) & \quad 0.6 \\
H) & \quad 0.7 \\
I) & \quad 0.8 \\
J) & \quad 0.9
\end{align*}
\]

\[
= \left( \int_{-1}^{1} e^{y \, d y} \right) \left( \int_{-1}^{1} \sin x \, d x \right) \left( \int_{-1}^{1} \, d x \right)
\]

\[
\text{since } \sin(x) \text{ is an odd function}
\]

\[
= 0
\]

14. Evaluate \( \iiint_B (x^2 + y^2 + z^2) \, dV \), where \( B \) is the unit ball centered at the origin.

\[
\begin{align*}
A) & \quad 2.50 \\
\mathcal{B} & \quad 2.51 \\
C) & \quad 2.52 \\
D) & \quad 2.53 \\
E) & \quad 2.54 \\
F) & \quad 2.55 \\
G) & \quad 2.56 \\
H) & \quad 2.57 \\
I) & \quad 2.58 \\
J) & \quad 2.59
\end{align*}
\]

\[
\begin{align*}
\mathcal{B} & \quad \frac{2 \pi}{5} \int_{0}^{\pi} \sin \phi \, d \phi \\
& \quad \left. \frac{2 \pi}{5} \left( \cos \phi \right) \right|_{0}^{\pi} \\
& \quad \frac{4 \pi}{5}
\end{align*}
\]
15. Let \( f \) and \( g \) be continuous functions on \( \mathbb{R}^3 \), and let \( E \) and \( F \) be ellipsoids with \( E \) contained in \( F \). Assume that \( f(x,y,z) \leq g(x,y,z) \) for all values of \((x,y,z)\). Which of the following 3 statements are true in general?

(I) \[ \iiint_E f(x,y,z) dV \leq \iiint_E g(x,y,z) dV \]

(II) \[ \iiint_E f(x,y,z) dV \leq \iiint_F f(x,y,z) dV \quad \text{take} \quad f = -1 \]

(III) \[ \iiint_E f(x,y,z) dV \leq \iiint_E f(x,y,z)^2 dV \quad \text{take} \quad f = \frac{1}{2} \]

Then:

A. All three statements are false.
B. Statement (I) is true, the other two are false.
C. Statement (II) is true, the other two are false.
D. Statement (III) is true, the other two are false.
E. Statement (I) is false, the other two are true.
F. Statement (II) is false, the other two are true.
G. Statement (III) is false, the other two are true.
H. All three statements are true.

16. What is the maximum value of

\[ f(x_1, \ldots, x_7) = (x_1 \ldots x_7)^{1/7} \]

where each \( x_i \) is positive and \( \sum_{i=1}^{7} x_i = 1 \)?

A) \( .10 \)
B) \( .11 \)
C) \( .12 \)
D) \( .13 \)
E) \( .14 \)
F) \( .15 \)
G) \( .16 \)
H) \( .17 \)
I) \( .18 \)
J) \( .19 \)

Let \( f(x_1, \ldots, x_7) = (x_1 \ldots x_7)^{1/7} \)

\[ g(x_1, \ldots, x_7) = x_1 + \ldots + x_7 \]

For any \( 1 \leq i < j \leq 7 \),

\[ \frac{1}{7} x_1^{-6/7} x_2^{1/7} \ldots x_7^{1/7} = \lambda \]

\[ \frac{1}{7} x_1 \ldots x_7 = \lambda \]

\[ \frac{1}{7} x_i = \frac{1}{7} \]

\[ \frac{1}{7} x_j = \frac{1}{7} \]

\[ \frac{1}{7} x_{ij} = \frac{1}{7} \]

\[ \frac{1}{7} x_1 = \ldots = x_7 = \lambda \]

\[ \therefore \text{max value of} \ f \text{ is} \ \frac{1}{7} \]