Final Exam

General Instructions: Read each problem carefully. Do only what is requested—nothing more nor less. Always show your work on each problem—unless the problem explicitly tells you not to show work. You will not get full credit for just writing down the answer. The point value of each problem is shown. Use the backs of the pages if you need more space to solve a problem.

The total time on this exam is two hours.
The total points on this exam is 200.
Ask questions if anything is unclear.

1. (10 points) Find the general solution to the differential equation

\[ xy' + y = 8. \]

\[ \frac{d}{dx} \left[ xy \right] = 8 \]

\[ xy = 8x + C \]

\[ y = 8 + \frac{C}{x} \]
2. (15 points) Give the general (implicit) solution to the exact differential equation

\[(\cos(x) + y) \, dx + (x + 3y^2) \, dy = 0.\]

Look for one function \(F(x, y)\) with

\[\frac{\partial F}{\partial x} = \cos(x) + y,\]
\[\frac{\partial F}{\partial y} = x + 3y^2.\]

\[\frac{\partial F}{\partial x} = \cos(x) + y\]
\[\downarrow\]
\[F = \sin(x) + xy + \Phi(y) \quad \rightarrow \quad F(x, y) = \sin(x) + xy + y^3\]

\[\frac{\partial F}{\partial y} = x + \Phi'(y) = x + 3y^2\]
\[\Phi'(y) = 3y^2\]

Choose \(\Phi(y) = y^3\)

\[\sin(x) + xy + y^3 = C\]

3. (10 points) Solve the initial value problem

\[y^2 \frac{dy}{dx} = \frac{2x}{3}\]
\[y^2y' = \frac{2x}{3}\]
\[y(3) = 1.\]

\[\int 3y^2 \, dy = \int 2x \, dx\]
\[y^3 = x^2 + C\]
\[1^3 = 3^2 + C\]
\[C = -8\]

\[y = x - 8\]

\[y = \sqrt[3]{x^2 - 8}\]
4. (10 points) (a) Find all the singular points for the following differential equation.

\[ y'' + \frac{1}{x^2 - 2x + 1} y' + \frac{x - 4}{x^2} y = 0 \]

\[ p(x) = \frac{1}{x^2 - 2x + 1} = \frac{1}{(x-1)^2} \]

\[ q(x) = \frac{x - 4}{x^2} \]

Singular points
\[ x = 1 \text{ and } x = 0 \]

(b) Find all the regular singular points for the same equation. Explain why they are regular singular points.

\[ (x-1) p(x) = \frac{i}{x-1} \quad \text{is not real analytic around } x = 1 \]

\[ x p(x) = \frac{x}{(x-1)^2} \]

\[ x^2 q(x) = x - 4 \]

\[ x = 0 \text{ is a regular singular point as } x p(x) \text{ and } x^2 q(x) \]

are real analytic around \( x = 0 \)

5. (10 points) Give the general solution to the differential equation

\[ y'' + 6y' + 8y = 0. \]

Guess \( y = e^{rx} \)

\[ y' = re^{rx} \]

\[ y'' = r^2 e^{rx} \]

\[ r^2 + 6r + 8 = 0 \]

\[ (r+4)(r+2) = 0 \]

\[ r = -4 \text{ or } r = -2 \]

\[ y = Ae^{-4x} + Be^{-2x} \]
6. (10 points)  Show that $y(x) = 3x^3 - 2x^2$ solves the following initial value problem.

$$y' = 9x^2 - 4x$$
$$y'' = 18x - 4$$

$$x^2 y'' - 4xy' + 6y = 0$$

$$y(1) = 3(1)^3 - 2(1)^2 = 1$$  \(\checkmark\)  
$$y'(1) = 9(1)^2 - 4(1) = 5$$  \(\checkmark\)

$$x^2 (18x - 4) - 4x (9x^2 - 4x) + 6(3x^2 - 2x^2)$$

$$= 18x^3 - 4x^2 - 36x^2 + 16x^2 + 18x^3 - 12x^2$$

$$= (18 - 36 + 18)x^3 + (-4 + 16 - 12)x^2$$

$$= 0$$  \(\checkmark\)

7. (15 points)  Use reduction of order to solve the differential equation

$$y = y'$$

$$yy'' = (y')^2$$

$$y \frac{dp}{dy} = p^2$$

$$\int \frac{dp}{p} = \int \frac{dy}{y}$$

$$\ln(p) = \ln(y) + C$$

$$p = e^{\ln(y) + C}$$

$$p = e^{C} e^{\ln(y)}$$

$$p = Ae$$

$$y = A e^{Ax}$$

$$y = Be^{Ax}$$
8. (15 points) Solve the following initial value problem.

\[ y'' - 4y = -6e^x \]
\[ y(0) = 4 \]
\[ y'(0) = 8 \]

General Homogeneous:  
\[ y'' - 4y = 0 \]
Guess  
\[ y = e^{rx} \]
\[ r^2 - 4 = 0 \]
\[ r = \pm 2 \]
\[ y = Ae^{2x} + Be^{-2x} \]

Particular Nonhomogeneous:  
\[ y'' - 4y = -6e^x \]
Guess  
\[ y = Ce^x \]
\[ Ce^x - 4Ce^x = -6e^x \]
\[ -3Ce^x = -6e^x \]
\[ c = 2 \]
\[ y = 2e^x \]

General Nonhomogeneous:  
\[ y = 2e^x + Ae^{2x} + Be^{-2x} \]
\[ y' = 2e^x + 2Ae^{2x} - 2Be^{-2x} \]
\[ y(0) = 4 \Rightarrow 2 + A + B = 4 \]
\[ A + B = 2 \]
\[ y'(0) = 8 \Rightarrow 2 + 2A - 2B = 8 \]
\[ 2A - 2B = 6 \]
\[ A - B = 3 \]
\[ \frac{A + B = 2}{A - B = 3} \]
\[ \frac{2A = 5}{A = 5/2} \]
\[ B = 2 - A = 2 - 5/2 = -1/2 \]

\[ y = 2e^x + \frac{5}{2}e^{2x} - \frac{1}{2}e^{-2x} \]
9. (15 points) Use power series methods to solve the equation \( y' - y = -2 \). (You do not need to recognize your solution as a closed form function. You may not solve the equation by other means and simply expand your answer into a power series, but you may check your answer this way.)

Guess \( y = \sum_{n=0}^{\infty} a_n x^n \)

\[ y' = \sum_{n=1}^{\infty} n a_n x^{n-1} \]

\[ -2 = y' - y = \sum_{n=1}^{\infty} n a_n x^{n-1} - \sum_{n=0}^{\infty} a_n x^n \]

\[ -2 = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n - \sum_{n=0}^{\infty} a_n x^n \]

\[ 0 = 2 + \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n - \sum_{n=0}^{\infty} a_n x^n \]

\[ 0 = (2 + a_1 - a_0) + \sum_{n=1}^{\infty} [(n+1) a_{n+1} - a_n] x^n \]

\[ 2 + a_1 - a_0 = 0 \]

\[ a_1 = a_0 - 2 \]

\[ (n+1) a_{n+1} - a_n = 0, \quad n \geq 1 \]

\[ a_{n+1} = \frac{a_n}{n+1}, \quad n \geq 1 \]

Let \( a_0 = c \), \( a_1 = c - 2 \), \( a_2 = \frac{c-2}{2} \), \( a_3 = \frac{c-2}{3 \cdot 2} \), \( \ldots \)

\[ a_n = \frac{c-2}{n!}, \quad n \geq 1 \]

\[ y = c + \sum_{n=1}^{\infty} \frac{c-2}{n!} x^n \]
10. (15 points) Calculate the inverse Laplace transform of \( G(p) = \frac{-6}{p^2 - 9} \).

We know \( \mathcal{L}^{-1} \left[ \frac{3}{p^2 - 9} \right] = \sinh(3x) \).

\[ \therefore \mathcal{L}^{-1} \left[ \frac{-6}{p^2 - 9} \right] = -2 \mathcal{L}^{-1} \left[ \frac{3}{p^2 - 9} \right] = -2 \sinh(3x) \]

(This could be expanded as \(-2 \cdot \frac{e^{3x} - e^{-3x}}{2} = e^{3x} - e^{-3x}\).

(See the end of the document for alternative solutions.)

11. (15 points) Find the eigenvalues \( \lambda_n \) and eigenfunctions \( y_n \) for the differential equation \( y'' + \lambda y = 0 \) and boundary conditions \( y(0) = 0, y(1) = 0 \).

The boundary conditions imply \( y \) takes the value 0 twice, so the chosen solution to \( y'' + \lambda y = 0 \) must have sine and cosine terms (not exponential terms).

\[ y'' + \lambda y = 0 \]

\[ \gamma^2 + \lambda = 0 \]

\[ \gamma = \pm \sqrt{-\lambda} \]

\[ \gamma = \pm \sqrt{-\lambda} \quad y(0) = 0 \Rightarrow A = 0, \quad y = B \sin(\sqrt{\lambda} x) \]

\[ y(1) = 0 \Rightarrow \sin(\sqrt{\lambda} \pi) = 0 \Rightarrow \sqrt{\lambda} = n \pi \]

Thus, the eigenvalues are \( \lambda_n = \frac{n^2 \pi^2}{9} \) and eigenfunctions \( y_n = \sin(n \pi x) \).
12. (10 points) Calculate the Laplace transform of \( f(x) = e^{2x} + x^2 \).

\[
L\left[ f(x) \right] = L\left[ e^{2x} \right] + L\left[ x^2 \right] = \frac{1}{\rho - 2} + \frac{2!}{\rho^{2+1}}
\]

\[
= \frac{1}{\rho - 2} + \frac{2}{\rho^3}
\]

13. (20 points) Let \( f(x) \) be defined as follows for parts (a)-(c).

\[
f(x) = \begin{cases} 
-1 & \text{if } -\pi \leq x < 0 \\
0 & \text{if } x = 0 \\
1 & \text{if } 0 < x \leq \pi
\end{cases}
\]

(a) Find the Fourier series for \( f(x) \) on \([-\pi, \pi] \). (Hint: \( f \) is odd.)

\[
f \text{ is odd } \Rightarrow a_j = 0
\]

\[
b_j = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(jx) \, dx = \frac{2}{\pi} \int_{0}^{\pi} f(x) \sin(jx) \, dx = \frac{2}{\pi} \int_{0}^{\pi} \sin(jx) \, dx
\]

Product is even

\[
= \frac{2}{\pi} \left[ \frac{-\cos(jx)}{j} \right]_{0}^{\pi} = \frac{2}{\pi} \left( \frac{-\cos(\pi j)}{j} + \frac{\cos(0)}{j} \right)
\]

\[
= j \frac{2}{\pi} \left( (-1)^{(j+1)} + 1 \right) = \begin{cases} 
\frac{4}{j\pi} & \text{if } j \text{ odd} \\
0 & \text{if } j \text{ even}
\end{cases}
\]

\[
f \sim \frac{4}{\pi} \sin(x) + \frac{4}{3\pi} \sin(3x) + \frac{4}{5\pi} \sin(5x) + \cdots
\]
(b) State the solution to the Dirichlet problem on the unit disk for boundary data $f(\theta)$ for $-\pi \leq \theta \leq \pi$ ($f$ is defined on the previous page). In the answer below, $r$ and $\theta$ are polar coordinates.

$$u(r, \theta) = \frac{4}{\pi} r \sin(\theta) + \frac{1}{3\pi} r^3 \sin(3\theta) + \frac{1}{5\pi} r^5 \sin(5\theta) + \ldots$$

(c) A vibrating string satisfies the wave equation $y_{xx} = y_{tt}$ with conditions $y(0,t) = 0$, $y(\pi, t) = 0$, and $\frac{\partial y}{\partial t}(x,0) = 0$. State the solution which also satisfies the initial condition $y(x,0) = 1$, $0 < x < \pi$. (Hint: think about how the initial condition on $(0, \pi)$ relates to the function $f$ defined on the previous page.)

$$y(x,t) = \frac{4}{\pi} \sin(x) \cos(t) + \frac{1}{3\pi} \sin(3x) \cos(3t) + \frac{1}{5\pi} \sin(5x) \cos(5t) + \ldots$$

In problems 14 to 19, circle the correct True/False responses (no work needed).

14. (5 points)  

\begin{itemize}
  \item True/False: The radius of converge of $\sum_{n=1}^{\infty} x^n$ is 1.
\end{itemize}

15. (5 points)  

\begin{itemize}
  \item True/False: The Laplace transform satisfies the identity $L[f \cdot g] = L[f] \cdot L[g]$.
\end{itemize}

16. (5 points)  

\begin{itemize}
  \item True/False: If $f(x)$ is an even function on $[-\pi, \pi]$, then the Fourier series of $f$ is of the form $\frac{a_0}{2} + \sum_{j=1}^{\infty} a_j \cos(jx)$.
\end{itemize}

17. (5 points)  

\begin{itemize}
  \item True/False: If $u_1$ and $u_2$ are solutions to the heat equation $u_{xx} = u_t$, then $u_1 + u_2$ is also a solution.
\end{itemize}

18. (5 points)  

\begin{itemize}
  \item True/False: If the Fourier series of $f(x)$ converges to 2 at $x = 0$, then the Cesaro means also converge to 2 at $x = 0$.
\end{itemize}

19. (5 points)  

\begin{itemize}
  \item True/False: The Laplace transform of $f(x) = \sin(x)$ is $F(p) = \frac{p}{p^2 + 1}$.
\end{itemize}
Alternative solutions to #10, \[ L^{-1} \left[ \frac{-6}{p^2 - q} \right] \]

\[
\frac{-6}{p^2 - q} = \frac{-b}{(p+3)(p-3)} = \frac{A}{p+3} + \frac{B}{p-3}
\]

\[-b = A(p-3) + B(p+3)\]

\[0p - 6 = (A+B)p + (3B-3A)\]

\[A + B = 0 \quad 3B - 3A = -6\]

\[\Rightarrow \quad B - A = -2\]

\[2B = -2 \quad A + (-1) = 0\]

\[B = -1 \quad A = 1\]

\[L^{-1} \left[ \frac{-6}{p^2 - q} \right] = L^{-1} \left[ \frac{1}{p+3} \right] - L^{-1} \left[ \frac{1}{p-3} \right] = \begin{bmatrix} -3x & 3x \\ \end{bmatrix} \]

\[L^{-1} \left[ \frac{-6}{p^2 - q} \right] = L^{-1} \left[ \frac{1}{p+3} \right] = -6 \quad L^{-1} \left[ \frac{1}{p-3} \right] = -6 \]

\[= -6 \cdot L^{-1} \left[ \frac{1}{p+3} \right] * L^{-1} \left[ \frac{1}{p-3} \right] = -6 \ e^{-3x} * e^{3x}\]

Note: convolution, not multiplication,

\[-6 \ e^{-3x} * e^{3x} = -6 \int_0^x e^{-3(x-t)} e^{3t} \ dt = -6 \ e^{-3x} \int_0^x e^{6t} \ dt\]

\[= -6 \ e^{-3x} \left( \frac{e^{6t}}{6} \right) \bigg|_{t=0}^{t=x} = -6 \ e^{-3x} \left( \frac{e^{6x}}{6} - \frac{1}{6} \right) = -e + e^{-3x}\]

\[= \frac{-3x}{e} + e^{-3x}\]

\[= e - e^3\]