Math 217 Exam 1    Sept 17, 2015

Instructions:

1. There are three parts in this exam. Part I is multiple choice, Part II is True/False, and Part III consists of hand-graded problems.

2. The total number of points is 100.

3. You may use a calculator.

4. The scorecard and Part III will be collected at the end of the exam. You may take Part I and Part II with you at the end of the exam.
Part I. Multiple Choices \quad 5 \times 10 = 50 \text{ points}

1. The direction field below corresponds which differential equation?

\[ A. \quad y' = \frac{x}{y} \]
\[ B. \quad y' = x^2 + y^2 \]
\[ C. \quad y' = -\frac{x}{y} \]
\[ D. \quad y' = x + y \]
\[ E. \quad y' = x - y \]
\[ F. \quad y' = x^2 - y^2 \]

A
2. Classify the differential equation \((\sin t)y'' + ty^2 = e^t\).

A. ordinary, linear, order 1
B. ordinary, linear, order 2
C. ordinary, non-linear, order 1
D. ordinary, non-linear, order 2
E. partial, linear, order 1
F. partial, non-linear, order 2

D
3. Which of the following is a solution of the differential equation \( y' = y + 2e^{-t} \)?

A. \( y(t) = e^{-t} \)
B. \( y(t) = \sin t \)
C. \( y(t) = e^t + e^{-t} \)
D. \( y(t) = e^t - e^{-t} \)
E. \( y(t) = \cos t \)
F. none of the above

D. If \( y = e^t - e^{-t} \),

then \( y' = e^t + e^{-t} \)

\[ y' = e^t + e^{-t} \]

\[ y + 2e^{-t} = e^t - e^{-t} + 2e^t = e^t + e^{-t} \]

so \( y' = y + 2e^{-t} \)
4. The differential equation

\[
\frac{xy'}{y} + e^y y' + \cos x + \ln y = 0
\]

belongs to only one of the following categories. Which one is it?

A. 1st order linear
B. separable
C. exact
D. 2nd order linear
E. 2nd order non-linear
F. autonomous

C

\[
\left( \frac{x}{y} + e^y \right) y' + \left( \cos x + \ln y \right) = 0
\]

\[
\frac{\partial}{\partial x} \left( \frac{x}{y} + e^y \right) = \frac{1}{y}
\]

\[
\frac{\partial}{\partial y} \left( \cos x + \ln y \right) = \frac{1}{y}
\]

so it is an exact equation.
5. Solve the initial value problem

\[ y' + 3t^2y = 0, \quad y(0) = 7. \]

A. \( y(t) = e^{-t} \)
B. \( y(t) = 7e^{-t} \)
C. \( y(t) = \frac{7}{2} (e^{-t^2} + e^{t^2}) \)
D. \( y(t) = 7e^{3t^2} \)
E. \( y(t) = 7e^{-t^3} \)
F. none of the above

\[ E \]

\[ \mu(t) = e \int 3t^2 dt = e^{t^3} \]

\[ \frac{d}{dt} (e^{t^3} y) = 0 \]

\[ e^{t^3} y = c. \]

\[ y = ce^{-t^3} \]

\[ y(0) = 7 \text{ implies } c = 7. \]

\[ y = 7e^{-t^3} \]

You may also solve it as a separable equation.
6. If $\phi$ is a solution of $y' = 6t(y - 1)^{\frac{2}{3}}$ and $\phi(0) = 1$, what is $\phi(1)$?

A. 0  
B. 1  
C. 2  
D. 9  
E. 33  
F. none of the above

Both B and C are correct, so you will get credit for this problem no matter what you choose.

\[
\frac{dy}{dt} = 6t(y - 1)^{\frac{2}{3}}
\]

**Case 1.** $y = 1$ is a solution so B is correct

**Case 2.** $(y-1)^{-\frac{1}{2}}dy = 6t\ dt$

\[
3(y-1)^{\frac{1}{2}} = 3t^2 + C
\]

$\phi(0) = 1 \Rightarrow C = 0$.

\[
3(\phi(1) - 1)^{\frac{1}{2}} = 3 \times 1^2 = 3.
\]

$\phi(1) = 2$.

So C is correct.
7. Determine the interval in which the solution exists for the initial value problem

\[ y' + y^3 = 0, \quad y(0) = 1. \]

A. \((-\infty, -\frac{1}{2})\)
B. \((-\frac{1}{2}, \infty)\)
C. \((-\infty, 0)\)
D. \((0, \infty)\)
E. \((-\frac{1}{2}, \frac{1}{2})\)
F. none of the above

\[ \frac{dy}{dt} = -y^3 \]

\[ y^{-3} \, dy = -dt \]

\[ -\frac{1}{2} y^{-2} = -t + C \]

\[ y(0) = 1 \quad \Rightarrow \quad C = -\frac{1}{2} \]

\[ -\frac{1}{2} y^{-2} = -t - \frac{1}{2} \]

\[ y^2 = \frac{1}{2t+1} \]

\[ y = \pm \sqrt{\frac{1}{2t+1}} \]

Using \( y(0) = 1 \), we find \( y(t) = \frac{1}{\sqrt{2t+1}} \).

Clearly, \( 2t+1 > 0 \).

\[ t > -\frac{1}{2} \]
8. Only one of the statements is true for the direction field below. Which one is it?

A. $y(t) = 0$ is a stable equilibrium solution.
B. $y(t) = 0$ is an unstable equilibrium solution.
C. $y(t) = 1$ is a stable equilibrium solution.
D. $y(t) = 1$ is an unstable equilibrium solution.
E. Neither $y(t) = 0$ nor $y(t) = 1$ are equilibrium solutions.
F. None of the above.

B
Find an integrating factor \( \mu \) that will make the following equation exact.

\[
y + 2tyy' = e^{-2y}y'
\]

A. \( \mu = e^{2y - \ln y} \)
B. \( \mu = \frac{1 - 2y}{2ty - e^{-2y}} \)
C. \( \mu = e^{2t - \ln t} \)
D. \( \mu = e^{\frac{3t}{2}} \)
E. \( \mu = e^{-\frac{1}{2y}} \)
F. None of the above.

A

\[
(2ty - e^{-2y})y' + y = 0
\]

\[
M = y, \quad N = 2ty - e^{-2y}
\]

\[
\frac{\frac{\partial N}{\partial t} - \frac{\partial M}{\partial y}}{M} = \frac{2y - 1}{y} = 2 - \frac{1}{y}.
\]

\[
\mu(y) = e^{\int \left(2 - \frac{1}{y}\right) dy} = e^{2y - \ln y}
\]
10. Which of the initial value problem below DOES NOT have a unique solution?

A. \( y' = 0, \quad y(0) = 0 \)
B. \( y' = y^{\frac{1}{2}}, \quad y(0) = 0 \)
C. \( y' = y, \quad y(0) = 0 \)
D. \( y' = y^2, \quad y(0) = 0 \)
E. \( y' = y^{2015}, \quad y(0) = 0 \)
F. None of the above.

\[
\frac{d\left(y^{\frac{1}{2}}\right)}{dy} = \frac{1}{2} y^{-\frac{1}{2}} \text{ is not continuous at } 0.
\]
Part II. True/False \hspace{1cm} 5 \times 2 = 10 \text{ points}

Choose 'A' if the statement is true; choose 'B' if the statement is false.

11. Only 1st order differential equations can be solved.

12. The function \( y(t) = -2t \) is an equilibrium solution to the differential equation \( y' = y + 2t \).

13. Consider the autonomous equation \( \frac{dy}{dt} = f(y) \), where \( f \) is a continuous function. It is NOT possible to have two stable equilibrium solutions with no other equilibrium solution between them.

14. The function \( \mu(t) = e^{2t} \) is an integrating factor for the equation

\[ ty' + 2y + e^t = 0. \]

15. There is only one solution to the initial value problem:

\[ y' = 5t^2, \quad y(1) = 2\pi. \]

B B A B A

12. \( y = -2t \) is NOT a solution.

13. Draw a picture!

Alternatively, the statement can be proven using the intermediate value theorem.

14. \( y' + \frac{2}{t}y + \frac{e^t}{t} = 0 \).

\[ \mu(t) = e^2 \int \frac{1}{t} \, dt = e^{2\ln(t)} + C. \]

for example, \( \mu = e^{2\ln(t)} = t^2 \) is an integrating factor.

15. Yes, by Uniqueness and Existence theorems.
Part III. Hand-graded problems \quad 10 + 10 + 20 = 40 points

16. (10 points)
Solve the initial value problem below.

\[ \frac{dy}{dt} + 2y = \sin t, \quad y\left(\frac{\pi}{2}\right) = 0. \]

\[ y' + \frac{2}{t} y = \frac{\sin t}{t} \]

\[ e^{\int \frac{2}{t} dt} = e^{2\ln t} = t^2. \]

Say \( e^{\int \frac{2}{t} dt} = t^2. \)

\[ \frac{d}{dt}(t^2 y) = t^2 \sin t \]

\[ t^2 y = \int t^2 \sin t \, dt \]

\[ = \int t (-\cos t)' \, dt \]

\[ = t(-\cos t) - \int (-\cos t) \, dt \]

\[ = -t \cos t + \sin t + C \]

\[ y(t) = -\frac{\cos t}{t} + \frac{\sin t}{t^2} + \frac{C}{t^2}. \]

\[ y\left(\frac{\pi}{2}\right) = 0 \Rightarrow 0 = 0 + \frac{1}{\left(\frac{\pi}{2}\right)^2} + \frac{C}{\left(\frac{\pi}{2}\right)^2} \Rightarrow C = -1. \]

\[ y(t) = -\frac{\cos t}{t} + \frac{\sin t}{t^2} - \frac{1}{t^2}. \]
17. (10 points)
Solve the differential equation below.

\[
\frac{dy}{dx} = \frac{x^2 - y^2}{x^2 - xy}
\]

Hint: Try the substitution \( v = \frac{y}{x} \).

\[
v = \frac{y}{x}
\Rightarrow y = xv \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}.
\]

\[
\frac{dy}{dx} = \frac{x^2 - y^2}{x^2 - xy} = \frac{1 - \left(\frac{y}{x}\right)^2}{1 - \frac{y}{x}}.
\]

\[
v + x\frac{dv}{dx} = \frac{1 - v^2}{1 - v}
\]

\[
x\frac{dv}{dx} = \frac{1 - v^2}{1 - v} - v = 1 + v - v = 1.
\]

\[
dv = \frac{dx}{x}.
\]

\[
v = \ln x + C
\]

\[
\frac{y}{x} = \ln x + C
\]

\[
y = x \ln x + Cx.
\]
16. \((10 + 10 = 20\) points\)

The radiation of a black body is governed by the Stefan-Boltzmann law. From it, we could derive a model for the variation of the temperature of a body with respect to its surroundings. The model is described by the differential equation

\[
\frac{du}{dt} = -\alpha \left( u^4 - T^4 \right)
\]

where \(u(t)\) is the temperature of the body at time \(t\) measured in Kelvin, \(T\) is the ambient temperature which we keep constant, and \(\alpha\) is an constant.

(a) Solve the differential equation.

Note: An implicit solution is good enough. You need to do a rather difficult integration.

\[
\frac{1}{u^4 - T^4} \, du = -\alpha \, dt
\]

\[
\frac{1}{(u^2+T^2)(u^2-T^2)} \, du = -\alpha \, dt
\]

**Partial fraction:**

\[
\frac{1}{2T^2} \left( -\frac{1}{u^2+T^2} + \frac{1}{u^2-T^2} \right) \, du = -\alpha \, dt
\]

**Partial fraction again:**

\[
-\frac{1}{2T^2} \frac{1}{u^2+T^2} + \frac{1}{2T} \left( -\frac{1}{u+T} + \frac{1}{u-T} \right) \, du = -\alpha \, dt
\]

Integrate both sides:

\[
-\frac{1}{2T^3} \arctan \left( \frac{u}{T} \right) + \frac{1}{4T^3} \ln \left( \frac{u+T}{u-T} \right) = -\alpha t + C
\]

\[
-\frac{1}{2T^3} \arctan \left( \frac{u}{T} \right) + \frac{1}{4T^3} \ln \left( \frac{u-T}{u+T} \right) = -\alpha t + C.
\]
(b) Now assume that the object is in vacuum, i.e. $T = 0K$. Then the model is simplified to be

$$\frac{du}{dt} = -\alpha u^4$$

Let us say $\alpha = 2.0 \times 10^{-10} K^{-3}/s$, the initial temperature $u(0) = 300K$. Calculate the time for the temperature of the body to reach 150K. (Round it to the nearest second.)

$$\frac{du}{u^4} = -\alpha dt$$

$$-\frac{1}{3u^3} = -\alpha t + C$$

$u(0) = 300 \Rightarrow C = -\frac{1}{3 \times 300^3}$

$$-\frac{1}{3 \times 150^3} = -2 \times 10^{-10} t - \frac{1}{3 \times 300^3}$$

$t \approx 432$ sec

So it takes 7 min 12 sec to cool down to 150K.