Instructions:

1. There are three parts in this exam. Part I is multiple choice, Part II is True/False, and Part III consists of hand-graded problems.

2. The total number of points is 100.

3. You may use a calculator.

4. The scantron and Part III will be collected at the end of the exam. You may take Part I and Part II with you at the end of the exam.

Here are some Taylor series that might be useful:

\[ e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \ldots \]

\[ \cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \ldots \]

\[ \sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots \]

\[ \cosh x = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} = 1 + \frac{x^2}{2} + \frac{x^4}{4!} + \ldots \]

\[ \sinh x = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots \]

\[ \ln(1 + x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots \]

\[ \frac{1}{1 - x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \ldots \]
Part I. Multiple Choices \[ 5 \times 10 = 50 \text{ points} \]

1. Consider power series \[
\sum_{n=0}^{\infty} \frac{n(n-1)}{2} x^n.
\]
From the power series below, choose the one that is different from the above.

A. \[
\sum_{n=1}^{\infty} \frac{n(n-1)}{2} x^n
\]

B. \[
\sum_{n=2}^{\infty} \frac{n(n-1)}{2} x^n
\]

C. \[
\frac{1}{2} \sum_{n=0}^{\infty} (n-1)^2 x^n + \frac{1}{2} \sum_{n=1}^{\infty} (n-1)x^n
\]

D. \[
\frac{1}{2} \sum_{n=0}^{\infty} n^2 x^{n+1} + \frac{1}{2} \sum_{n=1}^{\infty} (n-1)x^n
\]

E. \[
\frac{1}{2} \sum_{n=0}^{\infty} (n+2)^2 x^{n+2} - \frac{1}{2} \sum_{n=1}^{\infty} (n+1)x^{n+1}
\]

F. none of the above

C

The power series in C has an extra \( \frac{1}{2} \) from the first sum.
Clearly, A and B are the same as the original sum. For C, D and E, we rewrite the original sum as follows:

\[
\sum_{n=0}^{\infty} \frac{n(n-1)}{2} x^n = \sum_{n=0}^{\infty} \frac{(n-1)^2}{2} x^n + \sum_{n=0}^{\infty} \frac{n-1}{2} x^n
\]
\[
= \sum_{n=0}^{\infty} \frac{n^2}{2} x^n - \sum_{n=0}^{\infty} \frac{n}{2} x^n.
\]

Now we may shift the indices to obtain the forms in C, D and E. It turns out that the first sum in C should begin with \( n = 1 \), which means that the answer in C has an extra \( \frac{1}{2} \).
2. For initial value problem:

\[(x^2 - 1)y'' + (x + 1)y' - 2e^{x^2 - 1}y = 0, \quad y(0) = 0, \quad y'(0) = -2,\]

if \(y = \sum_{n=0}^{\infty} a_n x^n\) is the power series solution about 0, then we have...

A. \(a_0 = 0\) and \(a_1 = -2\)
B. \(a_0 = -2\) and \(a_1 = 0\)
C. \(a_0 = 0\) and \(a_1 = -\frac{1}{2}\)
D. \(a_0 = -\frac{1}{2}\) and \(a_1 = 0\)
E. \(a_0 = -2\) and \(a_1 = -2\)
F. none of the above

**A**

If \(y = \sum_{n=0}^{\infty} a_n x^n\), then

\[y(0) = a_0 = 0\]

and

\[y'(0) = a_1 = -2.\]
3. For differential equation:

\[(x^2 - 1)y'' + (x + 1)y' - 2e^{x^2 - 1}y = 0,\]  

(1)

if \(y = \sum_{n=0}^{\infty} a_n x^n\) is the general power series solution about 0, then without calculating it explicitly, what is the lower bound of the radius of convergence?

A. 0  
B. 1  
C. \(\frac{\pi}{2}\)  
D. 2\pi  
E. \(\infty\)  
F. none of the above

B

Rewriting (1), we obtain

\[y'' + \frac{1}{x - 1}y' - \frac{2e^{x^2 - 1}}{x^2 - 1}y = 0,\]

Since \(\frac{1}{x - 1}\) is analytic for \(x < 1\) and \(\frac{2e^{x^2 - 1}}{x^2 - 1}\) is analytic for \(-1 < x < 1\), it follows that the general power series solution about 0 has radius of convergence at least 1.
4. If we write $\cos[\ln(1 + x)]$ as a power series about 0, that is,

$$\cos[\ln(1 + x)] = \sum_{n=0}^{\infty} a_n x^n,$$

then what is the value of $a_4$?

A. 0  
B. $-\frac{1}{3}$  
C. $-\frac{1}{8}$  
D. $\frac{1}{24}$  
E. $-\frac{5}{12}$  
F. none of the above

E.

$$\cos[\ln(1 + x)] = 1 - \frac{\ln^2(1 + x)}{2!} + \frac{\ln^4(1 + x)}{4!} - \ldots$$

$$= 1 - \frac{1}{2} \left( x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots \right)^2 + \frac{1}{24} (x - \ldots)^4 - \ldots$$

$$= 1 + \ldots - \frac{1}{2} \left( \frac{-x^2}{2} \right)^2 - \frac{1}{2} \cdot 2x \cdot \frac{x^3}{3} + \frac{1}{24} x^4 + \ldots$$

$$= 1 + \ldots + \left( -\frac{1}{8} - \frac{1}{3} + \frac{1}{24} \right) x^4 + \ldots$$

$$= 1 + \ldots - \frac{5}{12} x^4 + \ldots$$
5. Which one of the following functions is NOT a solution of the differential equation
\[ 4x^2y'' - 8xy' + 9y = 0, \quad x \neq 0. \] (2)

A. \( y(x) = e^{3 \frac{\ln |x|}{2}} \)
B. \( y(x) = |x|^{\frac{3}{2}} \ln |x| \)
C. \( y(x) = e^{1+\frac{3}{2} \ln |x|+\ln(\ln |x|)} \)
D. \( y(x) = \ln \left( \frac{3}{2} |x| \right) \cdot |x|^\frac{3}{2} \)
E. \( y(x) = \left[ e - \ln \left( |x|^\frac{3}{2} \right) \right] \cdot |x|^\frac{3}{2} \)
F. all of the above are solutions of (2)

F

The indicial equation of (2) is
\[ 4r(r - 1) - 8r + 9 = 0, \]
\[ 4r^2 - 12r + 9 = 0. \]

We have repeated roots
\[ r_1 = r_2 = \frac{3}{2}. \]

Hence the general solution is
\[ y(x) = (c_1 + c_2 \ln |x|)|x|^\frac{3}{2}. \]

Now you can verify that all of A, B, C, D and E are solutions.
6. For the initial value problem:

\[ 2x^2 y'' - 5xy' + 5y = 0, \quad y(1) = 0, \quad y'(1) = \frac{3}{2}, \]

find \( x_0 \) where \( y'(x_0) = 0 \).

A. 1  
B. \( \left( \frac{4}{25} \right)^{\frac{1}{3}} \)  
C. \( \frac{5}{2} \)  
D. \( \frac{3}{2} \)  
E. \( \ln 5 - \ln 2 \)  
F. none of the above

B

The indicial equation of (2) is

\[ 2r(r - 1) - 5r + 5 = 0, \quad 2r^2 - 7r + 5 = 0. \]

We have the roots

\[ r_1 = 1, \quad r_2 = \frac{5}{2}, \]

and the general solution

\[ y(x) = c_1 x + c_2 x^{\frac{5}{2}}. \]

The initial condition implies

\[ c_1 + c_2 = 0, \quad c_1 + \frac{5}{2} c_2 = \frac{3}{2}, \]

so we have \( c_1 = -1, \ c_2 = 1 \), and

\[ y(x) = -x + x^{\frac{5}{2}}. \]

Taking derivative, we get

\[ y'(x) = -1 + \frac{5}{2} x^{\frac{3}{2}}, \]

so it follows that \( x_0 = \left( \frac{4}{25} \right)^{\frac{1}{3}} \).
7. Find the solution for the initial value problem:

\[ x^2 y'' - xy' + 2y = 0, \quad y(1) = 1, \quad y'(1) = 1. \]

A. \( x + 1 \)
B. \( x \cos(\ln x) \)
C. \( x \cos(\ln x) + x \sin(\ln x) \)
D. \( x \cos(\ln x) - x \sin(\ln x) \)
E. \( x^{1+i} \)
F. none of the above

B

The indicial equation of (2) is

\[[r(r - 1) - r + 2 = 0, \quad r^2 - 2r + 2 = 0.\]

We have the roots

\[ r_{1,2} = 1 \pm i, \]

and the general solution

\[ y(x) = x [c_1 \cos(\ln x) + c_2 \sin(\ln x)]. \]

Taking derivative, we get

\[ y'(x) = c_1 \cos(\ln x) + c_2 \sin(\ln x) - c_1 \sin(\ln x) + c_2 \cos(\ln x) \]
\[ = (c_1 + c_2) \cos(\ln x) + (-c_1 + c_2) \sin(\ln x). \]

The initial condition implies

\[ c_1 = 1, \quad c_1 + c_2 = 1, \]

so we have \( c_1 = 1 \) and \( c_2 = 0. \)
8. For the differential equation

\[(1 - \cos x) \cdot y'' + (e^x - 1) \cdot y' + \frac{3}{2} y = 0, \quad (3)\]

0 is a regular singular point. Find the indicial equation about 0.

A. \(r^2 - r + \frac{3}{2} = 0\)
B. \(r^2 + 2r - 3 = 0\)
C. \(r^2 + r - 3 = 0\)
D. \(r^2 - 4r + 4 = 0\)
E. \(r^2 - 3r + 2 = 0\)
F. none of the above

Rewriting (3), we get

\[y'' + \frac{e^x - 1}{1 - \cos x} \cdot y' - \frac{3}{2 - 2 \cos x} \cdot y = 0.\]

It follows that

\[p_0 = \lim_{x \to 0} \frac{x(e^x - 1)}{1 - \cos x} = \lim_{x \to 0} \frac{e^x - 1 + x e^x}{\sin x} = \lim_{x \to 0} \frac{e^x + e^x + x e^x}{\cos x} = 2,\]

and

\[q_0 = \lim_{x \to 0} \frac{3x^2}{2 - 2 \cos x} = \lim_{x \to 0} \frac{6x}{2 \sin x} = \lim_{x \to 0} \frac{6}{2 \cos x} = 3.\]

Thus, the indicial equation is

\[r(r - 1) + 2r + 3 = 0, \quad r^2 + r + 3 = 0.\]
9. Consider the power series

\[ y_1(x) = \sum_{n=0}^{\infty} \frac{a_n}{n!} x^n, \]
\[ y_2(x) = \sum_{n=0}^{\infty} \frac{b_n}{n!} x^n. \]

What is the Wronskian at 0? That is, what is \( W(y_1, y_2)(0) \)?

A. \( a_0 b_1 - a_1 b_0 \)
B. \( a_1 b_0 - a_0 b_1 \)
C. \( \frac{a_0 b_0}{2} - \frac{a_1 b_1}{2} \)
D. \( \frac{a_0 b_0}{2} - \frac{a_1 b_1}{2} \)
E. \( \sum_{n=0}^{\infty} a_n b_{n+1} - a_{n+1} b_n \)
F. none of the above

A

In fact, we just need the first two terms of (4):

\[ y_1(x) = a_0 + a_1 x + \ldots, \]
\[ y_2(x) = b_0 + b_1 x + \ldots, \]

which implies that

\[ y_1(0) = a_0, \quad y_1'(0) = a_1, \]
\[ y_2(0) = b_0, \quad y_2'(0) = b_1. \]

Thus,

\[ W(y_1, y_2)(0) = \begin{vmatrix} a_0 & b_0 \\ a_1 & b_1 \end{vmatrix} = a_0 b_1 - a_1 b_0. \]
10. Consider the differential equation

\[ x^2 y'' + (6x + x^2)y' + xy = 0, \quad (5) \]

for which 0 is regular singular point. To solve (5) by power series, we should begin by finding the coefficients of a Frobenius series. Of the Frobenius series below, which one is the correct trial solution?

A. \[ \sum_{n=1}^{\infty} a_n x^n \]

B. \[ 1 + \sum_{n=1}^{\infty} a_n x^n \]

C. \[ \sum_{n=1}^{\infty} a_n x^{n-5} \]

D. \[ |x|^{-5} \left( 1 + \sum_{n=1}^{\infty} a_n x^n \right) \]

E. \[ |x|^{-5} \ln |x| \left( 1 + \sum_{n=1}^{\infty} a_n x^n \right) \]

F. none of the above

B

It is easy to see that \( p_0 = 6 \) and \( q_0 = 0 \), so the indicial equation is

\[ r(r - 1) + 6r = r^2 + 5r = 0. \]

The two roots are \( r_1 = 0 \) and \( r_2 = -5 \), where 0 is the bigger root. Now, we follow the recipe outlined in Theorem 5.6.1 of Boyce-DiPrima.
Part II. True/False 5 × 2 = 10 points

Choose 'A' if the statement is true; choose 'B' if the statement is false.

11. For a second order linear homogeneous ordinary differential equation with constant coefficients, there is no singular points.

12. For two convergent power series

\[ y_1 = \sum_{n=0}^{\infty} a_n x^n, \quad y_2 = \sum_{n=0}^{\infty} b_n x^n, \]

the Wronskian \( W(y_1, y_2) \) is never zero.

13. For a second order linear homogeneous ordinary differential equation, we can always find two linearly independent power series solutions about an ordinary point.

14. If \( x_0 \) is a regular singular point of the differential equation

\[ y'' + p(x)y' + q(x)y = 0, \]

and the indicial equation has real roots, then the equation has at least one Frobenius series solution about \( x_0 \).

15. For the power series

\[ \sum_{n=0}^{\infty} a_n x^n, \]

if the radius of convergence is \( \rho \) and \( |x_1| > \rho \), then the series

\[ \sum_{n=0}^{\infty} a_n (x_1)^n \]

does not converge.

A B A A A
Math 217 Exam 3  Nov 17, 2015

Part III will be collected separately. Please write your NAME and your STUDENT NUMBER.
Student Number:

Name:

For graders:
16.

17.

18

19.

Total:

Here are some Taylor series that might be useful:

\[ e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \ldots \]

\[ \cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \ldots \]

\[ \sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots \]

\[ \cosh x = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} = 1 + \frac{x^2}{2} + \frac{x^4}{4!} + \ldots \]

\[ \sinh x = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots \]

\[ \ln(1 + x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots \]

\[ \frac{1}{1 - x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \ldots \]
Part III. Hand-graded problems \hspace{3em} 10 + 10 + 20 = 40 points

16. (10 points)
For the initial value problem:
\[ y'' = y' + y, \quad y(0) = 0, \quad y'(0) = 1, \]
the point 0 is an ordinary point. Show that the power series solution about 0 is given by
\[ y = \sum_{n=0}^{\infty} \frac{f_n}{n!} x^n, \]
where \( \{f_n\}_{n=0}^{\infty} \) is the Fibonacci numbers defined by \( f_0 = 0, \ f_1 = 1, \ f_{n+2} = f_{n+1} + f_n \) for \( n \geq 0 \).

Writing
\[ y = \sum_{n=0}^{\infty} a_n x^n, \]
we have
\[ y' = \sum_{n=1}^{\infty} a_n n x^{n-1}, \quad y'' = \sum_{n=2}^{\infty} a_n n(n-1) x^{n-2}. \]
Substituting these back into the differential equation, we get
\[ \sum_{n=2}^{\infty} a_n n(n-1) x^{n-2} = \sum_{n=1}^{\infty} a_n n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n, \]
\[ \sum_{n=0}^{\infty} a_{n+2} (n+2)(n+1)x^n = \sum_{n=0}^{\infty} a_{n+1} (n+1)x^n + \sum_{n=0}^{\infty} a_n x^n, \]
which implies
\[ a_{n+2} = \frac{a_n}{(n+2)(n+1)} + \frac{a_{n+1}}{n+2}. \]
The initial condition implies
\[ a_0 = 0 = \frac{f_0}{0!}, \]
\[ a_1 = 1 = \frac{f_1}{1!}. \]
Assuming \( a_m = \frac{f_m}{m!} \) for \( m \leq n+1 \), we get that
\[ a_{n+2} = \frac{a_n}{(n+2)(n+1)} + \frac{a_{n+1}}{n+2} = \frac{f_n}{n!} \frac{1}{(n+2)(n+1)} + \frac{f_{n+1}}{(n+1)!} \frac{1}{n+2} = \frac{f_n + f_{n+1}}{(n+2)!} = \frac{f_{n+2}}{(n+2)!}. \]
17. (10 points)
The Hermite function

\[ H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} \left( e^{-x^2} \right) \]

is a polynomial of degree n. Compute \( H_4 \) explicitly.

Hint: It is difficult to expend \( e^{x^2} \) as a power series and to calculate it term by term. Try something else.

\[
H_4(x) = e^{x^2} \frac{d^4}{dx^4} \left( e^{-x^2} \right)
\]
\[
= e^{x^2} \frac{d^3}{dx^3} \left( -2xe^{-x^2} \right)
\]
\[
= e^{x^2} \frac{d^2}{dx^2} \left( (4x^2 - 2)e^{-x^2} \right)
\]
\[
= e^{x^2} \frac{d}{dx} \left[ (4x^2 - 2)(-2x + 8) e^{-x^2} \right]
\]
\[
= e^{x^2} \frac{d}{dx} \left( -8x^3 + 12x \right) e^{-x^2} \]
\[
= e^{x^2} \left[ (-8x^3 + 12x)(-2x) - 24x^2 + 12 \right] e^{-x^2}
\]
\[
= 16x^4 - 48x^2 + 12.
\]
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For the differential equation

\[ 2x^2y'' + 3xy' - (x^2 + 1)y = 0, \]  

the point 0 is a regular singular point.

(a) Show that the roots of the indicial equation are \( r_1 = \frac{1}{2} \) and \( r_2 = -1 \).

Rewriting (6), we get

\[ y'' + \frac{3}{2x} \cdot y' - \frac{x^2 + 1}{2x^2} \cdot y = 0. \]

It follows that

\[ xp(x) = x \cdot \frac{3}{2x} = \frac{3}{2}, \]
\[ xq(x) = x^2 \cdot \left(-\frac{x^2 + 1}{2x^2}\right) = -\frac{1}{2} \cdot \frac{x^2}{2}. \]

Thus, the indicial equation is

\[ r(r - 1) + \frac{3}{2} r - \frac{1}{2} = 0, \]
\[ 2r^2 + r - 1 = 0 \]

where the roots are \( r_1 = \frac{1}{2} \) and \( r_2 = -1 \).
(b) Consider the Frobenius series solution

\[ y(x) = |x|^r \sum_{n=0}^{\infty} a_n x^n. \]

Show that the recurrence relation is

\[ a_n = \frac{a_{n-2}}{2(n + r)^2 + (n + r) - 1}, \quad n \geq 2, \]

and \( a_{2n+1} = 0 \).

For simplicity, we assume \( x > 0 \) to drop the absolute value. We begin with

\[ y(x) = \sum_{n=0}^{\infty} a_n x^{n+r}, \]
\[ y'(x) = \sum_{n=0}^{\infty} a_n (n + r)x^{n+r-1}, \]
\[ y''(x) = \sum_{n=0}^{\infty} a_n (n + r)(n + r - 1)x^{n+r-2}. \]

Substituting into (6), we get

\[ 2 \sum_{n=0}^{\infty} a_n(n + r)(n + r - 1)x^{n+r} + 3 \sum_{n=0}^{\infty} a_n(n + r)x^{n+r} - \sum_{n=0}^{\infty} a_n x^{n+r+2} - \sum_{n=0}^{\infty} a_n x^{n+r} = 0, \]
\[ 2 \sum_{n=0}^{\infty} a_n(n + r)(n + r - 1)x^{n+r} + 3 \sum_{n=0}^{\infty} a_n(n + r)x^{n+r} - \sum_{n=2}^{\infty} a_{n-2} x^{n+r} - \sum_{n=0}^{\infty} a_n x^{n+r} = 0, \]
\[ [2r(r-1) + 3r - 1]a_0 x^r + [2(r+1)r + 3(r+1) - 1]a_1 x^{r+1} \]
\[ + \sum_{n=0}^{\infty} \{[2(n + r)(n + r - 1) + 3(n + r) - 1]a_n - a_{n-2}\} x^{n+r} = 0, \]
\[ (2r^2 + r - 1)a_0 x^r + (2r^2 + 5r + 2)a_1 x^{r+1} + \sum_{n=0}^{\infty} \{[2(n + r)^2 + (n + r) - 1]a_n - a_{n-2}\} x^{n+r} = 0. \]

Thus,

\[ (2r^2 + r - 1)a_0 = 0, \]
\[ (2r^2 + 5r + 2)a_1 = 0, \]
\[ [2(n + r)^2 + (n + r) - 1]a_n - a_{n-2} = 0, \quad n \geq 2, \]

and the recurrence relation

\[ a_n = \frac{a_{n-2}}{2(n + r)^2 + (n + r) - 1}, \quad n \geq 2 \]

follows. Since \( a_0 \neq 0 \), we have the indicial equation \( 2r^2 + r - 1 = 0 \). For both \( r_1 = \frac{1}{2} \) and \( r_2 = -1 \), we have

\[ 2r^2 + 5r + 2 \neq 0. \]

It follows that \( a_1 = 0 \), and therefore \( a_{2n+1} = 0 \).
(c) For \( r_1 = \frac{1}{2} \), compute the Frobenius solution \( y_1 \) up to \( |x|^{\frac{1}{2}}x^6 \).

For \( r_1 = \frac{1}{2} \), the recurrence relation is

\[
a_n = \frac{a_{n-2}}{2(n + \frac{1}{2})^2 + (n + \frac{1}{2}) - 1} = \frac{a_{n-2}}{n(2n + 3)}, \quad n \geq 2.
\]

Thus,

\[
a_2 = \frac{a_0}{2 \cdot 7} = \frac{a_0}{14},
\]

\[
a_4 = \frac{a_2}{4 \cdot 11} = \frac{a_0}{616},
\]

\[
a_6 = \frac{a_4}{6 \cdot 15} = \frac{a_0}{55440},
\]

and it follows that

\[
y_1(x) = a_0|x|^{\frac{1}{2}} \left(1 + \frac{x^2}{14} + \frac{x^4}{616} + \frac{x^6}{55440} + \ldots\right).
\]
(d) For \( r_2 = -1 \), compute the Frobenius solution \( y_2 \) up to \( |x|^{-1} x^6 \).

For \( r_2 = -1 \), to distinguish from the case \( r_1 = \frac{1}{2} \), we write \( b_n \) in place of \( a_n \), so the recurrence relation is

\[
b_n = \frac{b_{n-2}}{2 (n - 1)^2 + (n - 1) - 1} = \frac{b_{n-2}}{n(2n - 3)}, \quad n \geq 2.
\]

Thus,

\[
\begin{align*}
b_2 &= \frac{b_0}{2 \cdot 1} = \frac{b_0}{2}, \\
b_4 &= \frac{b_2}{4 \cdot 5} = \frac{b_0}{40}, \\
b_6 &= \frac{b_4}{6 \cdot 9} = \frac{b_0}{2160},
\end{align*}
\]

and it follows that

\[
y_2(x) = b_0 |x|^{-1} \left( 1 + \frac{x^2}{2} + \frac{x^4}{40} + \frac{x^6}{2160} + \ldots \right).
\]