I. Short answers (22 points)

1. (2 points each)

Give an example of each of the following:

a. A function \( f : \mathbb{R} \to \mathbb{R} \) which is discontinuous everywhere.

Let \( Q \) be the set of rational numbers, and define
\[
    f(x) = \begin{cases} 
        1 & \text{if } x \in Q \\
        0 & \text{if } x \notin Q 
    \end{cases}
\]

b. A function \( f : \mathbb{R} \to \mathbb{R} \) which is continuous except at 0.

\[
    f(x) = \begin{cases} 
        1 & \text{if } x = 0 \\
        0 & \text{if } x \neq 0 
    \end{cases}
\]

c. A continuous function \( f : \mathbb{R} \to \mathbb{R} \) which is differentiable except at 0.

\[
    f(x) = |x|
\]

d. A function \( f : \mathbb{R}^2 \to \mathbb{R} \) whose partial derivatives at \( 0 \in \mathbb{R}^2 \) exist, but is not differentiable at \( 0 \in \mathbb{R}^2 \).

\[
    f \left( \begin{array}{c} x \\ y \end{array} \right) = \sqrt{|xy|}
\]

e. A differentiable function \( f : \mathbb{R} \to \mathbb{R} \) such that the derivative of \( f \) is not continuous at 0.

\[
    f(x) = \begin{cases} 
        x^2 \sin \left( \frac{1}{x} \right) & \text{if } x \neq 0 \\
        0 & \text{if } x = 0 
    \end{cases}
\]

f. A non-constant function \( f : \mathbb{R}^2 \to \mathbb{R} \) which is harmonic.

\[
    f \left( \begin{array}{c} x \\ y \end{array} \right) = x + y
\]

2. (2 points each)

For each of the following statement, indicate whether it is true or false:

a. There exists a subset of \( \mathbb{R}^n \) which is both open and closed.

True, e.g. \( \emptyset \).

b. For a closed set \( X \subset \mathbb{R}^n \), a continuous function \( f : X \to \mathbb{R} \) is uniformly continuous.

False, e.g. \( f(x) = x^2 \).

c. Fix a constant \( c \) such that \( 0 < c < 1 \). A bounded function \( f : \mathbb{R}^n \to \mathbb{R}^n \) satisfying
\[
    \|f(x) - f(y)\| \leq c\|x - y\|
\]

has a fixed point.

True. If \( \|f(x)\| \leq r \), then apply the contraction mapping principle to the restriction \( f \) to the closed ball \( B(0, r) \).
d. For a continuous function, the preimage of a closed set is closed.
   True. See Exercise 2.3.13, which is part of your homework.

e. For a continuous function, the image of a compact set is compact.
   True.

II. Continuity and differentiability (38 points)

1. (10 points)
Show that if $X \subset \mathbb{R}^n$ is not closed, then there is a continuous function $f : X \to \mathbb{R}$ which is unbounded.

   If $X$ is not closed, then there exists a converging sequence of points $\{x_k\}$ of points in $X$ such that the limit $x_0 = \lim_{k \to \infty} x_k$ is not in $X$. We define
   
   $$f : X \to \mathbb{R}, \quad f(x) = \frac{1}{||x - x_0||}.$$

   Then $f$ is continuous, but $f$ is unbounded.

2. (4 points + 10 points = 14 points)
   a. Let $U \subset \mathbb{R}^n$ be an open set, and let $f : U \to \mathbb{R}^m$. For a point $a \in U$, state what it means for $f$ be differentiable at $a$.

   The function $f$ be differentiable at $a$, if there exists a linear map $Df(a) : \mathbb{R}^n \to \mathbb{R}^m$ such that
   $$\lim_{h \to 0} \frac{f(a + h) - f(a) - Df(a)(h)}{||h||} = 0.$$

   b. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function such that $|f(x)| \leq ||x||^2$. Show that $f$ is differentiable at $0 \in \mathbb{R}^n$.

   First, we note that $|f(0)| \leq ||0||^2 = 0$ implies $f(0) = 0$.

   Next, we show that the $Df(0) = 0$. For $h \neq 0$, we have
   $$-||h|| \leq \frac{f(h)}{||h||} \leq ||h||.$$

   Since
   $$\lim_{h \to 0} -||h|| = \lim_{h \to 0} ||h|| = 0,$$

   it follows from the squeeze principle that
   $$\lim_{h \to 0} \frac{f(h) - f(0)}{||h||} = \lim_{h \to 0} \frac{f(h)}{||h||} = 0.$$

3. (4 points + 10 points = 14 points)
   a. State the chain rule.

   Let $g : \mathbb{R}^n \to \mathbb{R}^m$ be differentiable at $a$, and let $f : \mathbb{R}^m \to \mathbb{R}^l$ be differentiable at $g(a)$. Then $f \circ g$ is differentiable at $a$, and
   $$D(f \circ g)(a) = Df(g(a))Dg(a).$$

   b. Let $g : \mathbb{R} \to \mathbb{R}$ be a continuous function. We define
   $$f : \mathbb{R}^3 \to \mathbb{R}, \quad f \left( \begin{array}{c} x \\ y \\ z \end{array} \right) = \int_{x^y}^{\sin(x \sin(y \sin z))} g(t)dt.$$
Compute the derivative of $f$.

Since $g$ is continuous, $f$ is $C^1$. We compute the partial derivatives:

$$
\frac{\partial f}{\partial x} = g(\sin(x\sin(y\sin z))) \cdot \cos(x\sin(y\sin z)) \cdot \sin(y\sin z) - g(x^y) \cdot yx^{y-1},
$$

$$
\frac{\partial f}{\partial y} = g(\sin(x\sin(y\sin z))) \cdot \cos(x\sin(y\sin z)) \cdot \cos(y\sin z) \cdot \sin z - g(x^y) \cdot x^y \log(|x|),
$$

$$
\frac{\partial f}{\partial z} = g(\sin(x\sin(y\sin z))) \cdot \cos(x\sin(y\sin z)) \cdot \cos(y\sin z) \cdot y \cos z,
$$

and so

$$
Df \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{pmatrix}
$$

III. Extrema (20 points)

1. (10 points + 10 points = 20 points)

Consider the function

$$
f : \mathbb{R}^2 \to \mathbb{R}, \quad f \begin{pmatrix} x \\ y \end{pmatrix} = (x - y)e^{-(x^2+y^2)}
$$

a. Find and classify the critical points of $f$.

We begin by computing the partial derivatives:

$$
\frac{\partial f}{\partial x} = e^{-(x^2+y^2)} + (x - y)e^{-(x^2+y^2)}(-2x) = e^{-(x^2+y^2)}(1 - 2x^2 + 2xy),
$$

$$
\frac{\partial f}{\partial y} = -e^{-(x^2+y^2)} + (x - y)e^{-(x^2+y^2)}(-2y) = e^{-(x^2+y^2)}(-1 + 2xy + 2y^2).
$$

Thus, $Df = 0$ implies that

$$
1 - 2x^2 + 2xy = -1 - 2xy + 2y^2 = 0. \quad (1)
$$

Solving (1), we obtain the two critical points:

$$
\begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \end{pmatrix}, \quad \begin{pmatrix} -\frac{1}{2} \\ \frac{1}{2} \end{pmatrix}.
$$

On the other hand, we compute the second order partial derivatives:

$$
\frac{\partial^2 f}{\partial x^2} = e^{-(x^2+y^2)}(-4x + 2y) + e^{-(x^2+y^2)}(1 - 2x^2 + 2xy)(-2x)
$$

$$
= e^{-(x^2+y^2)}(-6x + 2y + 4x^3 - 4x^2 y),
$$

$$
\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} = e^{-(x^2+y^2)}(2x) + e^{-(x^2+y^2)}(1 - 2x^2 + 2xy)(-2y)
$$

$$
= e^{-(x^2+y^2)}(2x - 2y + 4x^2 y - 4xy^2),
$$

$$
\frac{\partial^2 f}{\partial y^2} = e^{-(x^2+y^2)}(-2x + 4y) + e^{-(x^2+y^2)}(-1 - 2xy + 2y^2)(-2y)
$$

$$
= e^{-(x^2+y^2)}(-2x + 6y + 4xy^2 - 4y^3).
$$

Thus, we have

$$
D^2f \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \end{pmatrix} = \frac{1}{\sqrt{e}} \begin{pmatrix} -3 & 1 \\ 1 & -3 \end{pmatrix}.
$$
Its characteristic polynomial
\[
\frac{1}{e} \det \begin{pmatrix} -3 - t & 1 \\ 1 & -3 - t \end{pmatrix} = \frac{1}{e} (t^2 + 6t + 8)
\]
has negative eigenvalues, so \(D^2 f \left( \frac{1}{2} \right)\) is negative-definite and \(\left( \frac{1}{2} \right)\) is a local maximum.

Similarly, we have
\[
D^2 f \left( -\frac{1}{2} \right) = -D^2 f \left( \frac{1}{2} \right)
\]
which must be positive-definite, and so \(\left( -\frac{1}{2} \right)\) is a local minimum.

b. Find the global extrema of \(f\) on the closed disk \(x^2 + y^2 \leq 1\). Why does the global extrema exist?

Since \(f\) is continuous and the closed disk \(x^2 + y^2 \leq 1\) is compact, it follows that the global extrema of \(f\) on the closed disk exist. They either occur at the interior critical points:
\[
\left( \frac{1}{2} \right), \quad \left( -\frac{1}{2} \right),
\]
or the constrained critical points on the boundary circle \(x^2 + y^2 = 1\).

We define
\[
g \begin{pmatrix} x \\ y \end{pmatrix} = x^2 + y^2
\]
with derivative
\[
Dg \begin{pmatrix} x \\ y \end{pmatrix} = (2x, 2y).
\]

By the Lagrangian multiplier theorem, the constrained critical points satisfies
\[
e^{-(x^2+y^2)} \begin{pmatrix} 1 - 2x^2 + 2xy \\ -1 - 2xy + 2y^2 \end{pmatrix} = \lambda \begin{pmatrix} x \\ y \end{pmatrix},
\]
\[
e^{-(x^2+y^2)} \begin{pmatrix} 1 - 2x^2 + 2xy \\ -1 - 2xy + 2y^2 \end{pmatrix} = \lambda \begin{pmatrix} x \\ y \end{pmatrix},
\]
\[
x^2 + y^2 = 1. \tag{2}
\]

If \(\lambda = 0\), the first two equations of (2) yields
\[
x = -y, \quad 4x^2 = 1,
\]
which is inconsistent with the third equation of (2).

On the other hand, if \(\lambda \neq 0\), then by eliminating \(\lambda\), we obtain from the first two equations of (2) that \(x = -y\), which together with the third equation yields the two constrained critical points:
\[
\left( \frac{1}{\sqrt{2}} \right), \quad \left( -\frac{1}{\sqrt{2}} \right).
\]

By checking the values at the various interior critical points and constrained critical points, we see that
\[
f \left( \frac{1}{2} \right) = \frac{1}{\sqrt{e}}
\]
is the global maximum, and
\[
f \left( -\frac{1}{2} \right) = \frac{1}{\sqrt{e}}
\]
is the global minimum.

Note, for example

\[ f \left( \frac{\sqrt{1}}{\sqrt{2}} \right) = \frac{\sqrt{2}}{e} < \frac{1}{\sqrt{e}}. \]

III. Contraction mapping theorem and inverse function theorem (20 points)

1. (10 points)

Let \( f : \mathbb{R} \to \mathbb{R} \) be a differentiable function. Show that \( f \) is a contraction map if and only if for all \( x \in \mathbb{R} \), we have \( |f'(x)| \leq c \) for some \( c \) satisfying \( 0 < c < 1 \).

If \( f \) is a contraction map, then

\[ |f(x) - f(y)| \leq c|x - y| \]

for some \( c \) satisfying \( 0 < c < 1 \). Thus, we have

\[
|f'(x)| = \left| \lim_{y \to x} \frac{f(x) - f(y)}{x - y} \right| = \lim_{y \to x} \frac{|f(x) - f(y)|}{|x - y|} \leq c.
\]

Conversely, if for all \( z \in \mathbb{R} \), \( |f'(z)| \leq c \) for some \( c \) satisfying \( 0 < c < 1 \), then by the mean value theorem, for \( x < y \), there exists some \( z \) satisfying \( x < z < y \) such that

\[ f(x) - f(y) = f'(z)(x - y). \]

Therefore,

\[ |f(x) - f(y)| = |f'(z)||x - y| \leq c|x - y|. \]

2. (6 points + 4 points = 10 points)

a. Let \( f : \mathbb{R} \to \mathbb{R} \) be a differentiable function. Show that if for all \( x \in \mathbb{R} \),

\[ f'(x) \neq 0, \]

then \( f \) is 1-1 on all of \( \mathbb{R} \).

We prove by contradiction. Assume that \( f \) is not 1-1, i.e. there exist \( x_1, x_2 \in \mathbb{R} \) with \( x_1 < x_2 \) such that \( f(x_1) = f(x_2) \). Then by mean value theorem or Rolle’s theorem, there exists \( x \) satisfying \( x_1 < x < x_2 \) such that \( f'(x) = 0 \).

b. Show that the result in a. does not generalize to higher dimensions. That is, show that the following statement is false:

Let \( U \subset \mathbb{R}^n \) be an open set, and let \( f : U \to \mathbb{R}^n \) be a differentiable function. If for all \( x \in U \),

\[ \det Df(x) \neq 0, \]

then \( f \) is 1-1 on all of \( U \).

For example, we may take the function

\[ f : \mathbb{R}^2 \to \mathbb{R}^2, \quad f \left( \begin{array}{c} u \\ v \end{array} \right) = \left( \begin{array}{c} e^u \cos v \\ e^u \sin v \end{array} \right) \]

Although

\[ \det Df \left( \begin{array}{c} u \\ v \end{array} \right) = e^{2u} \neq 0, \]

it is clear that \( f \) is not 1-1, e.g.

\[ f \left( \begin{array}{c} u \\ v + 2\pi \end{array} \right) = f \left( \begin{array}{c} u \\ v \end{array} \right). \]