Ma 233: Calculus III
Solutions to Midterm Examination 2

Profs. Krishtal, Ravindra, and Wickerhauser
18 questions on 18 pages

Monday, March 14th, 2005

1. Find parametric equations for the tangent line at the point $(1/2, -\sqrt{3}/2, -\pi/3)$ on the curve $\mathbf{r}(t) = (\cos t, \sin t, t)$

(a) $(1/2, -\sqrt{3}/2, -\pi/3) + t \left(1/2, \sqrt{3}/2, 1\right)$

(b) $(1/2, -\sqrt{3}/2, -\pi/3) + t \left(\sqrt{3}/2, 1/2, 1\right)$

(c) $(1/2, -\sqrt{3}/2, -\pi/3) + t \left(-1/2, \sqrt{3}/2, 1\right)$

(d) $(1/2, -\sqrt{3}/2, -\pi/3) + t \left(-\sqrt{3}/2, 1/2, 1\right)$

(e) $(1/2, \sqrt{3}/2, 1) + t \left(1/2, -\sqrt{3}/2, -\pi/3\right)$

(f) $(\sqrt{3}/2, 1/2, 1) + t \left(1/2, -\sqrt{3}/2, -\pi/3\right)$

(g) $(-1/2, \sqrt{3}/2, 1) + t \left(1/2, -\sqrt{3}/2, -\pi/3\right)$

(h) $(-\sqrt{3}/2, 1/2, 1) + t \left(1/2, -\sqrt{3}/2, -\pi/3\right)$

Solution: Differentiate to get $\mathbf{r}'(t) = (-\sin t, \cos t, 1)$. The tangent line at $t = -\pi/3$ has direction vector $\mathbf{r}'(-\pi/3) = (-\sin(-\pi/3), \cos(-\pi/3), 1) = (\sqrt{3}/2, 1/2, 1)$ and base point $\mathbf{r}(-\pi/3) = (1/2, -\sqrt{3}/2, -\pi/3)$, so it may be written as

$$(1/2, -\sqrt{3}/2, -\pi/3) + t \left(\sqrt{3}/2, 1/2, 1\right)$$
The parameteric equations of the tangent line are thus

\[
\begin{align*}
x(t) &= \frac{1}{2} + \frac{\sqrt{3}}{2} t \\
y(t) &= -\frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2} t \\
z(t) &= -\frac{\pi}{3} + t
\end{align*}
\]
2. If \( \mathbf{r}(t) = \cos(7t) \mathbf{i} + \sin(-5t) \mathbf{j} + 6t \mathbf{k} \), compute \( \int_0^{\pi/2} \mathbf{r}(t) \, dt \).

(a) \( \frac{1}{7} \mathbf{i} - \frac{1}{5} \mathbf{j} + \frac{3\pi^2}{4} \mathbf{k} \).

(b) \( \frac{1}{7} \mathbf{i} + \frac{1}{5} \mathbf{j} + \frac{3\pi^2}{4} \mathbf{k} \).

(c) \( \frac{1}{7} \mathbf{i} - \frac{1}{5} \mathbf{j} - \frac{3\pi^2}{4} \mathbf{k} \).

(d) \( -\frac{1}{7} \mathbf{i} - \frac{1}{5} \mathbf{j} + \frac{3\pi^2}{4} \mathbf{k} \). \( \Leftarrow \)

(e) \( \mathbf{i} - \mathbf{j} + 3\pi \mathbf{k} \).

(f) \( \mathbf{i} + \mathbf{j} + 3\pi \mathbf{k} \).

(g) \( \mathbf{i} - \mathbf{j} - 3\pi \mathbf{k} \).

(h) \( -\mathbf{i} - \mathbf{j} + 3\pi \mathbf{k} \).

Solution: Integrate the components separately to get the antiderivative

\[
\frac{1}{7} \sin(7t) \mathbf{i} + \frac{1}{5} \cos(-5t) \mathbf{j} + \frac{6t^2}{2} \mathbf{k}
\]

Evaluating the difference between 0 and \( \pi/2 \) gives

\[
-\frac{1}{7} \mathbf{i} - \frac{1}{5} \mathbf{j} + \frac{3\pi^2}{4} \mathbf{k}
\]

\( \square \)
3. Find the length of the curve

\[ \{ \mathbf{r}(t) = \cos(5t)\mathbf{i} + \sin(5t)\mathbf{j} + 6t\mathbf{k} : -2 \leq t \leq 6 \} \]

(a) 8  
(b) 48  
(c) \(2\sqrt{61}\)  
(d) \(4\sqrt{61}\)  
(e) \(8\sqrt{61}\)  
(f) \(2\sqrt{86}\)  
(g) \(4\sqrt{86}\)  
(h) \(8\sqrt{86}\)

**Solution:** Find \( \mathbf{r}'(t) = -5\sin(5t)\mathbf{i} + 5\cos(5t)\mathbf{j} + 6\mathbf{k} \) and

\[
\| \mathbf{r}'(t) \| = \sqrt{25\sin^2(5t) + 25\cos^2(5t) + 36} = \sqrt{61}.
\]

Integrate this to get the length

\[
\int_{-2}^{6} \| \mathbf{r}'(t) \| \, dt = 8\sqrt{61}.
\]
4. Given that a point has acceleration $a(t) = (1, 2t, -3t + 1)$, its position is $(1, 1, 1)$ at $t = 0$ and its velocity is $(-2, -2, -2)$ at $t = 0$, find its position at all times $t$.

(a) $r(t) = \left( \frac{1}{2} t^2 - t + 2, \frac{1}{3} t^3 - t + 2, -\frac{1}{2} t^3 + \frac{1}{2} t^2 - t + 2 \right)$

(b) $r(t) = \left( \frac{1}{2} t^2 + t + 2, \frac{1}{3} t^3 + t + 2, -\frac{1}{2} t^3 + \frac{1}{2} t^2 + t + 2 \right)$

(c) $r(t) = \left( \frac{1}{2} t^2 - t - 2, \frac{1}{3} t^3 + t - 2, -\frac{1}{2} t^3 + \frac{1}{2} t^2 + t - 2 \right)$

(d) $r(t) = \left( \frac{1}{2} t^2 - t - 2, \frac{1}{3} t^3 - t - 2, -\frac{1}{2} t^3 + \frac{1}{2} t^2 - t - 2 \right)$

(e) $r(t) = \left( \frac{1}{2} t^2 - 2t + 1, \frac{1}{3} t^3 - 2t + 1, -\frac{1}{2} t^3 + \frac{1}{2} t^2 - 2t + 1 \right) \Leftarrow$

(f) $r(t) = \left( \frac{1}{2} t^2 + 2t + 1, \frac{1}{3} t^3 + 2t + 1, -\frac{1}{2} t^3 + \frac{1}{2} t^2 + 2t + 1 \right)$

(g) $r(t) = \left( \frac{1}{2} t^2 + 2t - 1, \frac{1}{3} t^3 + 2t - 1, -\frac{1}{2} t^3 + \frac{1}{2} t^2 + 2t - 1 \right)$

(h) $r(t) = \left( \frac{1}{2} t^2 - 2t - 1, \frac{1}{3} t^3 - 2t - 1, -\frac{1}{2} t^3 + \frac{1}{2} t^2 - 2t - 1 \right)$

Solution: Velocity, the antiderivative of acceleration, is

$$v(t) = \left( t + c_x, \ t^2 + c_y, \ -\frac{3}{2} t^2 + t + c_z \right),$$

where $c_x, c_y, c_z$ are constants of integration. Determine these from the condition $(−2, −2, −2) = v(0) = (c_x, c_y, c_z)$, so $c_x = −2, c_y = −2, c_z = −2$ and $v(t) = \left( t - 2, \ t^2 - 2, \ -\frac{3}{2} t^2 + t - 2 \right)$.

Position, the antiderivative of velocity, is

$$r(t) = \left( \frac{1}{2} t^2 - 2t + k_x, \ \frac{1}{3} t^3 - 2t + k_y, \ -\frac{1}{2} t^3 + \frac{1}{2} t^2 - 2t + k_z \right),$$

where $k_x, k_y, k_z$ are constants of integration. Determine these from the condition $(1, 1, 1) = r(0) = (k_x, k_y, k_z)$, so $k_x = 1, k_y = 1, k_z = 1$ and

$$r(t) = \left( \frac{1}{2} t^2 - 2t + 1, \ \frac{1}{3} t^3 - 2t + 1, \ -\frac{1}{2} t^3 + \frac{1}{2} t^2 - 2t + 1 \right)$$

$\square$
5. Find the curvature of \( y = \cos(3x) \) at \( x = \pi/4 \)

(a) 3/2
(b) 2/3
(c) 11/13^{2/3}
(d) 11/13^{3/2}
(e) 13/11^{2/3}
(f) 13/18^{3/2}
(g) 18/11^{2/3}
(h) 18/11^{3/2}  \Leftarrow
(i) 11/18^{3/2}
(j) 13/11^{3/2}

**Solution:** This is the special case of a curve being the graph of a function: \( y = f(x) = \cos(3x) \). Therefore, use the formula

\[
\kappa(x) = \frac{|f''(x)|}{(1 + [f'(x)]^2)^{3/2}}
\]

But \( f'(x) = -3\sin(3x) \) and \( f''(x) = -9\cos(3x) \), so the formula expands to

\[
\kappa(x) = \frac{|9\cos(3x)|}{(1 + 9\sin^2(3x))^{3/2}}
\]

Thus

\[
\kappa(\pi/4) = \frac{|-9/\sqrt{2}|}{(1 + 9/2)^{3/2}} = \frac{18}{11^{3/2}}
\]
6. Let \( \mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + 2t\mathbf{k} \). Find the unit binormal vector \( \mathbf{B}(t) \) for all \( t \).

(a) \((-\sin t)\mathbf{i} + (\cos t)\mathbf{j} + 2\mathbf{k}\)
(b) \((\sin t)\mathbf{i} + (-\cos t)\mathbf{j} + 2\mathbf{k}\)
(c) \(-\frac{\sin t}{\sqrt{5}}\mathbf{i} + \frac{\cos t}{\sqrt{5}}\mathbf{j} + \frac{2}{\sqrt{5}}\mathbf{k}\)
(d) \(\frac{\sin t}{\sqrt{5}}\mathbf{i} + \frac{\cos t}{\sqrt{5}}\mathbf{j} + \frac{2}{\sqrt{5}}\mathbf{k}\)
(e) \(-\frac{\cos t}{\sqrt{5}}\mathbf{i} + \frac{\sin t}{\sqrt{5}}\mathbf{j} + 0\mathbf{k}\)
(f) \(-\frac{\cos t}{\sqrt{5}}\mathbf{i} + \frac{\sin t}{\sqrt{5}}\mathbf{j} + 0\mathbf{k}\)
(g) \(-\cos(t)\mathbf{i} - \sin(t)\mathbf{j} + 0\mathbf{k}\)
(h) \(\cos(t)\mathbf{i} + \sin(t)\mathbf{j} + 0\mathbf{k}\)
(i) \(-\frac{2\sin t}{\sqrt{5}}\mathbf{i} + \frac{-2\cos t}{\sqrt{5}}\mathbf{j} + \frac{1}{\sqrt{5}}\mathbf{k}\)
(j) \(\frac{2\sin t}{\sqrt{5}}\mathbf{i} + \frac{-2\cos t}{\sqrt{5}}\mathbf{j} + \frac{1}{\sqrt{5}}\mathbf{k}\)

\[\iff\]

**Solution:** Tangent vector: \( \mathbf{r}'(t) = (-\sin t)\mathbf{i} + (\cos t)\mathbf{j} + 2\mathbf{k} \). Its length is the constant 

\[\|\mathbf{r}'(t)\| = \sqrt{\sin^2 t + \cos^2 t + 4} = \sqrt{5} \text{ for all } t, \text{ so the unit tangent vector is} \]

\[ \mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = \frac{-\sin t}{\sqrt{5}}\mathbf{i} + \frac{\cos t}{\sqrt{5}}\mathbf{j} + \frac{2}{\sqrt{5}}\mathbf{k}. \]

Normal vector:

\[ \mathbf{T}'(t) = \frac{-\cos t}{\sqrt{5}}\mathbf{i} + \frac{-\sin t}{\sqrt{5}}\mathbf{j} + 0\mathbf{k}, \]

and its length is

\[ \|\mathbf{T}'(t)\| = \sqrt{(-\cos t)^2/5 + (\sin t)^2/5} = 1/\sqrt{5}. \]

Hence the unit normal vector is

\[ \mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|} = -\cos(t)\mathbf{i} - \sin(t)\mathbf{j} + 0\mathbf{k}. \]

Binormal vector:

\[ \mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t) = \frac{2\sin t}{\sqrt{5}}\mathbf{i} + \frac{-2\cos t}{\sqrt{5}}\mathbf{j} + \frac{1}{\sqrt{5}}\mathbf{k} \]

\[\square\]
7. Match the following surfaces with the verbal description of their level curves:

1. \[ z = \sqrt{9-x^2-y^2} \]
2. \[ z = y^2 - x^2 \]
3. \[ z = \frac{1}{y} - 3 \]

X. a collection of parallel lines
Y. a collection of circles
Z. two lines and a collection of hyperbolas

(a) 1 is X; 2 is Y; 3 is Z.
(b) 1 is Y; 2 is Z; 3 is X. ⇐
(c) 1 is Z; 2 is X; 3 is Y.
(d) 1 is X; 2 is Z; 3 is Y.
(e) 1 is Z; 2 is Y; 3 is X.
(f) 1 is Y; 2 is X; 3 is Z.
(g) All of them are X.
(h) All of them are Y.
(i) All of them are Z.
(j) None of the above.

**Solution:** Equation 1 is a hemisphere, with circles (Y) as level sets. Equation 2 is a hyperbolic paraboloid, with lines and hyperbolas (Z) as level sets. Equation 3 is hyperbolic cylinder with parallel lines (X) as level sets. □
8. Find \( \lim_{(x,y) \to (0,0)} \frac{x^2 - 3xy + y^2}{(x - y)^2} \), if it exists.

(a) \(-4\)
(b) \(-3\)
(c) \(-2\)
(d) \(-1\)
(e) 0
(f) 1
(g) 2
(h) 3
(i) 4
(j) The limit does not exist. \(\Leftarrow\)

**Solution:** The limit does not exist. Along the line \(y = 0\) through \((0,0)\), the ratio is constantly 1, while along the line \(y = -x\) through \((0,0)\), the ratio is constantly \(\frac{5}{4}\). Since these do not agree at \((0,0)\), there can be no limit at \((0,0)\). \(\Box\)
9. Find the partial derivative \( f_{xxy} \) for the function \( f(x, y) = e^{xy^2} \).

(a) \( y^6 e^{xy^2} \)

(b) \( 4y^3 e^{xy^2} \)

(c) \( 4y^3 e^{xy^2} + 2y^5 e^{xy^2} \)

(d) \( 4y^3 e^{xy^2} + 2xy^5 e^{xy^2} \)

(e) \( 2ye^{xy^2} + 3xy^5 e^{xy^2} \)

(f) \( 2ye^{xy^2} + 2xy^3 e^{xy^2} \)

(g) \( 3xy^3 e^{xy^2} \)

(h) \( y^5 xe^{xy^2} \)

(i) The derivative exists, but is none of the above.

(j) The derivative does not exist.

Solution: \( f_x = y^2 e^{xy^2}; \ f_{xx} = y^4 e^{xy^2}; \ f_{xxy} = 4y^3 e^{xy^2} + 2xy^5 e^{xy^2}; \)
10. Determine whether each of the following functions is a solution to Laplace’s equation

\[ u_{xx} + u_{yy} = 0. \]

(I) \( u(x, y) = x^2 + y^2 \)

(II) \( u(x, y) = x^2 - y^2 \)

(III) \( u(x, y) = \log \sqrt{x^2 + y^2} \)

(a) I only.

(b) II only.

(c) III only.

(d) IV only.

(e) I and II only.

(f) I and III only.

(g) II and III only. \( \Leftarrow \)

(h) All.

(i) None.

**Solution:**

For (I): \( u_x = 2x, \ u_{xx} = 2; \ u_y = 2y, \ u_{yy} = 2. \) Thus \( u_{xx} + u_{yy} = 4 \neq 0. \)

For (II): \( u_x = 2x, \ u_{xx} = 2; \ u_y = -2y, \ u_{yy} = -2. \) Thus \( u_{xx} + u_{yy} = 0. \)

For (III): \( u_x = x/(x^2 + y^2), \ u_{xx} = (y^2 - x^2)/(x^2 + y^2)^2; \ u_y = y/(x^2 + y^2), \ u_{yy} = (x^2 - y^2)/(x^2 + y^2)^2; \) Thus \( u_{xx} + u_{yy} = 0. \) \( \square \)
11. Find the equation of the tangent plane to the surface $z = x^2 - y^2$ at the point $(2, 1, 3)$.

(a) $2x - y - z = 0$
(b) $2x - y - z = 3$
(c) $4x - 2y - z = 0$
(d) $4x - 2y - z = 3\quad \Leftarrow$
(e) $4x - 2y - z = 6$
(f) $8x - 4y - 2z = 3$
(g) $x - 2y - \frac{1}{2}z = 3$
(h) $x - 2y + \frac{1}{2}z = 3$
(i) $x - 2y - \frac{1}{2}z = 6$
(j) $x - 2y + \frac{1}{2}z = 6$

**Solution:** This surface is a level surface for the function $F(x, y, z) = x^2 - y^2 - z = 0$, so its tangent plane at $(2, 1, 3)$ has $\nabla F(2, 1, 3) = \langle 2x, -2y, -1 \rangle \big|_{(2,1,3)} = \langle 4, -2, -1 \rangle$ as a normal vector. Since $(2, 1, 3)$ is in the tangent plane, its equation is $4x - 2y - z = (4, -2, -1) \cdot \langle 2, 1, 3 \rangle = 3$. \[\square\]
12. Use differentials to estimate the amount of metal in a closed cylindrical can that is 10 cm high and 4 cm in diameter if the metal in the top and bottom is 0.1 cm thick and the metal in the sides is 0.05 cm thick.

(a) 8.0 cm$^3$
(b) 8.1 cm$^3$
(c) 8.2 cm$^3$
(d) 8.3 cm$^3$
(e) 8.4 cm$^3$
(f) 8.5 cm$^3$
(g) 8.6 cm$^3$
(h) 8.7 cm$^3$
(i) 8.8 cm$^3$ ←
(j) 8.9 cm$^3$

**Solution:** Let $h = 10$ cm denote the height of the can, and $d = 4$ cm denote its diameter. The area of the top and bottom together is $2 \times \pi (d/2)^2 = 8 \pi \approx 25$, while the area of the side is $h \times \pi d = 40 \pi \approx 126$, so the total metal in the can is approximately $(0.10)(25) + (0.05)(126) = 8.8$
13. Let \( z = f(x - y) \) for a differentiable function \( f \). Then \( \partial z/\partial x + \partial z/\partial y \) is

(a) 1
(b) 2
(c) \( \sqrt{5}/2 \)
(d) \( \pi \)
(e) \( \infty \)
(f) 0 \( \Leftarrow \)
(g) \( -1 \)
(h) \( -2 \)

**Solution:** By the chain rule, \( \partial z/\partial x = f'(x - y) \) while \( \partial z/\partial y = (-1)f'(x - y) \). Therefore, \( \partial z/\partial x + \partial z/\partial y = 0 \). □
14. Find the directional derivative of \( g(x, y, z) = 3e^x \cos(yz) \) at the point \( P(0, 0, 0) \) in the direction \( \langle \frac{2}{3}, \frac{1}{3}, -\frac{2}{3} \rangle \).

(a) 2  \quad \leftrightarrow \quad \mathbf{b}
(b) 1
(c) -3
(d) 5
(e) 0
(f) -4
(g) \infty
(h) 3.43

**Solution:** Compute the gradient at \( P(0, 0, 0) \):

\[
\nabla g(0, 0, 0) = (3e^x \cos(yz), -3ze^x \sin(yz), -3ye^x \sin(yz)) \bigg|_{(0,0,0)} = \langle 3, 0, 0 \rangle .
\]

Evaluate the dot product to get the directional derivative in the direction \( \mathbf{u} = \langle \frac{2}{3}, \frac{1}{3}, -\frac{2}{3} \rangle \):

\[
D_{\mathbf{u}} g(0, 0, 0) = \nabla g(0, 0, 0) \cdot \mathbf{u} = \langle 3, 0, 0 \rangle \cdot \langle \frac{2}{3}, \frac{1}{3}, -\frac{2}{3} \rangle = 2.
\]

\[\square\]
15. Consider the surface $7x^2 - 3y^2 + z^2 = 8$. Find an equation for the plane tangent to this surface at its point $P(1, 1, 2)$.

(a) $7x - 3y + 2z = 8$  ⇐

(b) $7x - 3y + z = 8$

(c) $-3y + 8z = 10$

(d) $-3y + 5z = 16$

(e) $-3y + 4z = 20$

(f) $2x - z = 2$

(g) $x = y$

(h) $z = 0$

(i) None of the above

**Solution:** The surface is a level set for the function $f(x, y, z) = 7x^2 - 3y^2 + z^2$. This function is a polynomial, hence it is differentiable and its gradient $\nabla f(1, 1, 2)$ is a normal vector for the tangent plane at $P(1, 1, 2)$. But

$$\nabla f(1, 1, 2) = \langle 14x, -6y, 2z \rangle|_{(1,1,2)} = \langle 14, -6, 4 \rangle,$$

so the equation of the tangent plane is

$$\langle 14, -6, 4 \rangle \cdot \langle x, y, z \rangle = \langle 14, -6, 4 \rangle \cdot \langle 1, 1, 2 \rangle$$

which simplifies to the equation $7x - 3y + 2z = 8$.  

□
16. The volume of the largest rectangular box in the first octant with three faces in the coordinate planes and one vertex in the plane \( x + 2y + 3z = 6 \) is

(a) \( \frac{3}{4} \)
(b) \( 5 \)
(c) \( \frac{7}{2} \)
(d) \( \frac{4}{3} \)
(e) \( \frac{\sqrt{34}}{83} \)
(f) \( \sqrt{34} \)
(g) \( \sqrt{84} \)
(h) None of the above

**Solution:** Let \( x, y, z \) be the coordinates of the vertex on the plane \( x + 2y + 3z = 6 \). The dimensions of the box will then be \( x, y, z \) and its volume will be

\[ xyz = (6 - 2y - 3z)yz = f(y, z) = 6yz - 2y^2z - 3yz^2 \]

To maximize this function over all \( y, z \) that occur in the first octant, first find the critical points:

\[ \nabla f(y, z) = \langle 6z - 4yz - 3z^2, 6y - 2y^2 - 6yz \rangle = \langle 0, 0 \rangle \]

\[ \iff 6z - 4yz - 3z^2 = 0 \text{ and } 6y - 2y^2 - 6yz = 0 \]

\[ \iff z(6 - 4y - 3z) = 0 \text{ and } y(6 - 2y - 6z) = 0. \]

If either \( z = 0 \) or \( y = 0 \), then the volume of box will be 0, clearly the absolute minimum. This accounts for all points \( x, y, z \) on one of the coordinate planes, too; they are all absolute minima for the volume.
If both $z$ and $y$ are nonzero, then $\nabla f(y, z) = \langle 0, 0 \rangle$ only if $6 - 4y - 3z = 0$ and $6 - 2y - 6z = 0$. But this is a linear system of two equations in two unknowns:

\[
\begin{align*}
4y + 3z &= 6 \\
2y + 6z &= 6
\end{align*}
\]

Subtracting twice the second equation from the first and solving the result yields $z = 2/3$, so $y = 1$. This must be the unique absolute maximum, as it is a critical point and the positive value of $g(y, z)$ there exceeds the zero value at all the other critical points.

The corresponding third dimension of the box will be $x = 6 - 2y - 3z = 2$, so the box will have volume $(2)(1)(\frac{2}{3}) = \frac{4}{3}$. \qed
17. The points on the hyperboloid $x^2 - y^2 + 2z^2 = 1$ where the normal line is parallel to
the line that joins the points $(3, -1, 0)$ and $(5, 3, 6)$ are

(a) $(3, -1.0)$ and $(5, 3, 6)$

(b) $(\sqrt{35}, \sqrt{38}, \sqrt{2})$ and $(-\sqrt{35}, -\sqrt{38}, \sqrt{2})$

(c) $(2, 2, 2)$ and $(1, 1, 1)$

(d) $(\sqrt{6}, 3, -\sqrt{2})$ and $(\sqrt{6}, 3, \sqrt{2})$

(e) any two points.

(f) $(1, 0, 0)$ and $(0, 0, 1/\sqrt{2})$

(g) $(\sqrt{6}/3, -2\sqrt{6}/3, \sqrt{6}/2)$ and $(-\sqrt{6}/3, 2\sqrt{6}/3, -\sqrt{6}/2)$  $\Leftarrow$

(h) None of the above.

**Solution:** The normal line is given by the gradient of $f(x, y, z) = x^2 - y^2 + 2z^2$ for
which the hyperboloid is a level set. But

$$\nabla f(x, y, z) = (2x, -2y, 4z)$$

The line joining the points $(3, -1, 0)$ and $(5, 3, 6)$ has direction vector $\mathbf{u} = (3, -1, 0) - (5, 3, 6) = (2, 4, 6)$. The normal is parallel to this line if and only if $\nabla f(x, y, z) \times \mathbf{u} = 0$, or

$$\langle 2x, -2y, 4z \rangle \times \langle 2, 4, 6 \rangle = \langle 0, 0, 0 \rangle \quad \iff \quad \begin{cases} 8x + 4y = 0 \\ -12y - 16z = 0 \\ -12x + 8z = 0 \end{cases} \iff \begin{cases} 2x + y = 0 \\ 3y + 4z = 0 \\ 3x - 2z = 0 \end{cases}$$

Substituting $y = -2x$ from the first equation and $z = \frac{3}{2}x$ from the third equation into
the second equation yields $-6x + 6x = 0$, so the system is consistent and any points
on the parameterized line $\{x = t, y = -2t, z = \frac{3}{2}t\}$ will solve it. Substitute these
parametric formulas into the equation of the hyperboloid to find the desired point:

\[ 1 = f(t, -2t, \frac{3}{2}t) = (t)^2 - (-2t)^2 + 2\left(\frac{3}{2}t\right)^2 = \frac{3}{2}t^2, \]

so \( t = \pm\sqrt{\frac{2}{3}} \), giving the points \((x, y, z) = (\sqrt{\frac{2}{3}}, -2\sqrt{\frac{2}{3}}, \frac{3}{2}\sqrt{\frac{2}{3}}) = (\sqrt{6}/3, 2\sqrt{6}/3, \sqrt{6}/2)\)
and \((x, y, z) = (-\sqrt{6}/3, 2\sqrt{6}/3, -\sqrt{6}/2).\) \(\square\)
18. A flat circular plate has the shape of the region \( \{(x, y) : x^2 + y^2 \leq 1\} \). The plate is cooled so that the temperature at the point \((x, y)\) is

\[
T(x, y) = x^2 + 2y^2 - x
\]

The extreme temperatures on the surface are

(a) 3.17 and 2.16

(b) 2.25 and −0.25

(c) 4.33 and −4.33

(d) 5.15 and 0

(e) 0.91 and 0.19

(f) 1.67 and −3.20

(g) 3.33 and −4.04

(h) 0.1 and 0.1

**Solution:** Look for critical points: \( 0 = \nabla T(x, y) = \langle 2x - 1, 4y \rangle \) implies \( x = \frac{1}{2}, y = 0 \). There, the temperature is \( T(\frac{1}{2}, 0) = -0.25 \).

Look on the boundary. Substituting the circle parameterization \( x = \cos t, y = \sin t \) for the boundary points gives \( f(t) = T(\cos t, \sin t) = \cos^2 t + 2 \sin^2 t - \cos t \). This function is extremal at its critical points \( 0 = f'(t) = -2 \cos t \sin t + 4 \sin t \cos t + \sin t \), where

\[
\sin t(2 \cos t + 1) = 0 \quad \iff \quad y(2x + 1) = 0.
\]

This is true on the circle \( x^2 + y^2 = 1 \) if and only if either \( y = 0 \) and \( x = \pm 1 \), or \( x = -\frac{1}{2} \) so \( y = \pm \frac{\sqrt{3}}{2} \). The corresponding temperatures are:

\[
T(1, 0) = 0; \quad T(-1, 0) = 2; \quad T(-\frac{1}{2}, \frac{\sqrt{3}}{2})) = 2.25; \quad T(-\frac{1}{2}, -\frac{\sqrt{3}}{2})) = 2.25
\]

The extremal values from these five candidates are \( T = -0.25 \) and \( T = 2.25 \). \( \square \)