1. Prove that a metric space \( S \) is disconnected if and only if there is a subset \( A \subset S \) which is neither empty nor all of \( S \), but which is both open and closed in \( S \).

2. A set \( S \subset \mathbb{R}^n \) is called starlike if there is some base point \( x \in S \) such that for every point \( y \in S \), the line between \( x \) and \( y \) is contained in \( S \).
   (a) Prove that every convex subset of \( \mathbb{R}^n \) is starlike.
   (b) Prove that every starlike subset of \( \mathbb{R}^n \) is connected.

3. Prove that if \( f : \mathbb{R} \to \mathbb{R} \) is continuous and one-to-one on a compact interval \( [a, b] \), then \( f \) must be strictly monotonic on \( [a, b] \).

4. A function \( f : \mathbb{R} \to \mathbb{R} \) is said to satisfy a Lipschitz condition of order \( \alpha > 0 \) at a point \( c \) in its domain if
   \[
   \exists M > 0 \quad \exists r > 0 \quad \forall x \in B(c; r), x \neq c \quad |f(x) - f(c)| < M|x - c|^{\alpha}.
   \]
   (a) Prove that if \( f \) satisfies a Lipschitz condition of order \( \alpha > 0 \) at \( c \), then \( f \) is continuous at \( c \).
   (b) Prove that if \( f \) satisfies a Lipschitz condition of order \( \alpha > 1 \) at \( c \), then \( f \) is differentiable at \( c \).
   (c) Find a function \( f \) satisfying a Lipschitz condition of order \( \alpha = 1 \) at \( c \) but which is not differentiable at \( c \).

5. Suppose that \( f \) is defined on \((0, 1)\) and has a bounded derivative in \((0, 1)\) (i.e., there is a finite \( M > 0 \) such that \(|f'(x)| \leq M \) for all \( x \in (0, 1) \)). Put \( a_n = f(1/n) \) for \( n = 1, 2, 3, \ldots \). Prove that \( \lim_{n \to \infty} a_n \) exists. (Hint: use the Cauchy criterion.)

6. Let \( f \) be continuous on \([0, 1]\) with \( f(0) = 0 \) and \( f'(x) \) finite at each \( x \in (0, 1) \). Suppose \( f'(x) \) is an increasing function on \((0, 1)\). Prove that \( g(x) \defeq f(x)/x \) is an increasing function on \((0, 1)\).

7. Prove that if \( f \) has a finite third derivative \( f''' \) in \([a, b]\), and \( f(a) = f(b) = f'(a) = f'(b) = 0 \), then there must be some point \( c \in (a, b) \) for which \( f'''(c) = 0 \).

8. Suppose that the vector-valued function \( \mathbf{x} \) is differentiable at each point \( t \in (a, b) \), and that \( ||\mathbf{x}|| \) is constant on \((a, b)\). Prove that \( \mathbf{x}(t) \cdot \mathbf{x}'(t) = 0 \) for all \( t \in (a, b) \).

9. Define a real-valued function \( f \) of two real variables as follows:
   \[
   f(x, y) = \frac{xy}{x^2 + y^2}, \quad (x, y) \neq 0; \quad f(0, 0) = 0.
   \]
   (a) Prove that the partial derivatives \( D_1 f(x, y) \) and \( D_2 f(x, y) \) exist for every \((x, y) \in \mathbb{R}^2\) and find explicit formulas for them.
   (b) Show that \( f \) is not continuous at \((0, 0)\).
10. Let $S$ be an open set in $\mathbb{C}$ and let $S^*$ be the set of complex conjugates of points of $S$: $S^* \overset{\text{def}}{=} \{ \bar{z} : z \in S \}$. If $f$ is defined on $S$, define $g$ on $S^*$ by the formula $g(z) = \overline{f(z)}$. Prove that if $f$ is differentiable at $c \in \text{int} S$, then $g$ is differentiable at $\bar{c}$. 