1. Determine (with proof) whether the function \( f(x) \) defined as \( f(x) = \frac{x^{1/5} \cos(\pi/2x)}{\pi/2x} \) if \( x \neq 0 \), with \( f(0) = 0 \), has bounded variation on the interval \([-1, 1]\).

2. A function \( \mu = \mu(x) \) defined on \( \mathbb{R}^+ \) is called a Marcinkiewicz multiplier if there is some \( M < \infty \) such that \( V_\mu(2^j, 2^{j+1}) < M \) for all integers \( j \); that is, \( \mu \) has uniformly bounded variation on intervals of the form \([2^j, 2^{j+1})\).
   (a) Prove that \( \mu(x) = \log x \) is a Marcinkiewicz multiplier. Thus such functions do not have to be bounded.
   (b) Prove that \( \mu \) is a Marcinkiewicz multiplier if and only if there is some \( \lambda > 1 \) and some \( N < \infty \) such that \( V_\mu(a, \lambda a) < N \) for every \( a > 0 \).

3. A real-valued function \( f \) defined on \([a, b] \subset \mathbb{R}\) is said to absolutely continuous on \([a, b]\) if for every \( \epsilon > 0 \) there is \( \delta > 0 \) such that for every finite collection of disjoint open subintervals \((a_i, b_i) \subset [a, b]\) with \( \sum |b_i - a_i| < \delta \), we have \( \sum |f(b_i) - f(a_i)| < \epsilon \).
   Prove that a function which is absolutely continuous on \([a, b]\) is continuous and of bounded variation on \([a, b]\).

4. Suppose that \( x \) is a rectifiable path of length \( L \) defined on \([a, b]\) and assume that \( x \) is not constant on any subinterval of \([a, b]\). Let \( s(x) = \Lambda_x(a, x) \) if \( a < x \leq b \) and put \( s(a) = 0 \). Prove that \( s^{-1} \) exists and is continuous on \([0, L]\).

5. Give an example of a bounded function \( f \) and an increasing function \( \alpha \) defined on \([a, b]\) such that \( |f| \in R(\alpha) \) but \( f \notin R(\alpha) \).

6. Assume that \( \alpha \) has bounded variation on \([a, b]\) and \( f \in R(\alpha) \) on \([a, b]\). Let \( V(x) \) denote the total variation of \( \alpha \) on \([a, x]\), where \( a < x \leq b \), and put \( V(a) = 0 \) as usual. Show that
   \[
   \left| \int_a^b f \, d\alpha \right| \leq \int_a^b |f| \, dV \leq MV(b),
   \]
   where \( M = \sup\{|f(t)| : a \leq t \leq b\} \).

7. Let \( f \) be a positive continuous function in \([a, b]\). Let \( M \) denote the maximum value of \( f \) on \([a, b]\). Show that
   \[
   \lim_{n \to \infty} \left( \int_a^b f(x)^n \, dx \right)^{1/n} = M.
   \]
8. Assume that $f$ has a decreasing derivative which satisfied $f'(x) \geq m > 0$ for all $x \in [a, b]$. Prove that

$$\left| \int_a^b \cos f(x) \, dx \right| \leq \frac{2}{m}.$$  

(Hint: Multiply and divide the integrand by $f'(x)$ and use Theorem 7.37(ii)).

9. Prove that the following function is Riemann integrable on $[0, 1]$:

$$f(x) = \begin{cases} 1, & \text{if } x = 0; \\ 0, & \text{if } x \in (0, 1) \text{ is irrational}; \\ 1/n, & \text{if } x \in (0, 1] \text{ is rational, with } x = m/n \text{ in lowest terms}. \end{cases}$$

(Hint: compute the oscillation $\omega_f(x)$ of $f$ at each $x \in [0, 1]$.)

10. Define

$$g(x) = \begin{cases} 0, & \text{if } x = 0; \\ 1, & \text{if } 0 < x \leq 1. \end{cases}$$

(a) Prove that $g$ is Riemann integrable on $[0, 1]$.

(b) Let $f = f(x)$ be as in Problem 9. Prove that $g \circ f$ is not Riemann integrable on $[0, 1]$, despite both $f \in R$ and $g \in R$ on $[0, 1]$. 

2