1. Let $a_n = \frac{n!}{n^n}$ for $n > 0$. Prove that $\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \frac{1}{e}$.

Solution:

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)!/(n+1)^{n+1}}{n^n/n^n} = \frac{n^n(n+1)!}{(n+1)^{n+1}n!} = \frac{n^n(n+1)!}{(n+1)(n+1)^n} = \left(\frac{n}{n+1}\right)^n.$$  

The last expression equals $(1 + \frac{1}{n})^{-n}$, which tends to $\frac{1}{e}$ as $n \to \infty$.  

2. Suppose that $\{a_n\}$ is a bounded sequence of real numbers with the property that $\liminf_{n \to \infty} a_n \geq \limsup_{n \to \infty} a_n$. Prove that $\lim_{n \to \infty} a_n$ exists.

Solution: By theorem 8.3(a) of the text, $\liminf_{n \to \infty} a_n \leq \limsup_{n \to \infty} a_n$. From the hypothesis we conclude that $\liminf_{n \to \infty} a_n = \limsup_{n \to \infty} a_n$. Since $\{a_n\}$ is bounded both limits are finite, and thus by theorem 8.3(b) we conclude that that $\lim_{n \to \infty} a_n$ exists.

3. Suppose that $\sum_{n=1}^\infty a_n$ converges absolutely. Define $\{b_n\}$ by

$$b_n = \begin{cases} a_n^2, & \text{if } n \text{ is even}, \\ -a_n^2, & \text{if } n \text{ is odd}. \end{cases}$$

Prove that $\sum_{n=1}^\infty b_n$ converges absolutely.

Solution: Since $\sum |a_n|$ converges, we must have $\lim_{n \to \infty} |a_n| = 0$. Thus we can find an $N < \infty$, without loss of generality assuming $N > 2$, such that $n > N \Rightarrow |a_n| < 1 \Rightarrow |a_n^2| < |a_n|$ and $n > N \Rightarrow |a_n| < 1 \Rightarrow |a_n^2| < |a_n|$. Thus for all $n > N$, $0 \leq |b_n| \leq |a_n|$, and so the series $\sum |b_n|$ converges by theorem 8.20.

4. Determine with proof whether the following series converges:

$$\sum_{n=1}^\infty \left(\sqrt{1+n^6} - n^3\right)$$

Solution: Rewrite

$$\sum_{n=1}^\infty \left(\sqrt{1+n^6} - n^3\right) = \sum_{n=1}^\infty \frac{\left(\sqrt{1+n^6} - n^3\right) \left(\sqrt{1+n^6} + n^3\right)}{\sqrt{1+n^6} + n^3} = \sum_{n=1}^\infty \frac{1}{\sqrt{1+n^6} + n^3}.$$ 

This converges by comparison with $\sum \frac{1}{n^3}$.  

1
5. Determine with proof whether the following series converges:

$$\sum_{n=1}^{\infty} (\sqrt{1+n} - \sqrt{n})$$

**Solution:** Rewrite

$$\sum_{n=1}^{\infty} (\sqrt{1+n} - \sqrt{n}) = \sum_{n=1}^{\infty} \frac{(\sqrt{1+n} - \sqrt{n})(\sqrt{1+n} + \sqrt{n})}{\sqrt{1+n} + \sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{\sqrt{1+n} + \sqrt{n}}.$$  

This diverges by comparison with \(\sum \frac{1}{\sqrt{n}}\); the summands are larger than \(\frac{1}{2n}\) for all \(n > 0\).

Alternatively, notice that the series telescopes, so that for any \(M \geq 1\),

$$\sum_{n=1}^{M} (\sqrt{1+n} - \sqrt{n}) = \sqrt{M+1} - \sqrt{1}.$$  

This partial sum increases without bound as \(M \to \infty\), so the series diverges.

6. Determine with proof whether the following series converges:

$$\sum_{n=2}^{\infty} (\log n)^{-\log n}$$

**Solution:** We use the integral test, with the substitution \(x \leftarrow e^y, \; dx = e^y \; dy\):

$$\int_{e}^{\infty} (\log x)^{-\log x} \; dx = \int_{1}^{\infty} y^{-y} e^y \; dy = \int_{1}^{\infty} e^{y(1-\log y)} \; dy.$$  

This converges by comparison with \(\int_{1}^{\infty} e^{-y} \; dy < \infty\), since \(1 - \log y < -1\) for all \(y > e^2\).

7. Find a double sequence \(\{a_{n,m}\}\) such that \(\lim_{n \to \infty} a_{n,m} = 0\) for all fixed \(m\) and \(\lim_{m \to \infty} a_{n,m} = 0\) for all fixed \(n\), but \(\lim_{n,m \to \infty} a_{n,m}\) does not exist.

**Solution:** Consider \(a_{n,m} = \frac{1}{n^{-m+0.5}}\). For each fixed \(n\), \(\frac{1}{n^{-m+0.5}} \to 0\) as \(m \to \infty\). Likewise, for each fixed \(m\), \(\frac{1}{n^{-m+0.5}} \to 0\) as \(n \to \infty\). Hence \(a_{n,m}\) is the only candidate for the limit of the double sequence. However, \(a_{n,n} = 2\) no matter how large a value we take for \(n\).

8. Find the Cesàro sum of the complex-valued series \(\sum_{n=0}^{\infty} i^n\), where \(i^2 = -1\).

**Solution:** The partial sums \(s_m = \sum_{n=0}^{m} i^n\) take the values 1, \(1 + i\), \(i\), and 0, depending on whether the remainder left after dividing \(m\) by 4 is 0, 1, 2, or 3, respectively. Therefore, for any integer written \(m = 4k + m'\) \(\geq 0\) with \(0 \leq m' < 4\), we have

$$\sigma_m = s_0 + s_1 + \ldots + s_m = \frac{k(1 + [1 + i] + i + 0)}{4k + m'} + \frac{s_m}{4k + m'}.$$  

But since \(s_m/m \to 0\) as \(m \to \infty\), and \(k/(4k + m') \to \frac{1}{4}\) as \(m \to \infty\), the Cesàro sum is evidently \((1 + i)/2\).

9. Prove that \(\prod_{n=2}^{\infty} (1 - n^{-2})\) converges and evaluate it.

**Solution:** Convergence follows from theorem 8.52, since the series \(\sum n^{-2}\) converges. Rewriting the partial product yields \(\prod_{n=2}^{K} \frac{n^2-1}{n^2} = \prod_{n=2}^{K} \frac{(n-1)(n+1)}{n^2}\), wherein one factor of the numerator cancels part of a past denominator, while the other cancels part of a future denominator. Hence the product telescopes down to \(\left(\frac{2-1}{2}\right) \left(\frac{K+1}{K}\right)\), which tends to the limit \(\frac{1}{2}\) as \(K \to \infty\).
10. Prove that if a double series converges absolutely, then it converges.

**Solution:** Write \( \sum_{n,m} f(n, m) \) for the double series and \( \{s(p, q)\} \) for the double sequence of partial sums. We will show that \( s(p, q) \) satisfies the Cauchy condition, namely, that for any \( \epsilon > 0 \) we can find \( N < \infty \) such that \( |s(p_1, q_1) - s(p_2, q_2)| < \epsilon \) whenever \( p_1 > p_2 \geq N \) and \( q_1 > q_2 \geq N \). But this follows from the Cauchy condition which is satisfied by \( S(p, q) \), the sequence of partial sums of the series \( \sum |f(n, m)| \), and from the triangle inequality:

\[
|s(p_2, q_2) - s(p_1, q_1)| &= \left| \sum_{p_1+1}^{p_2} \sum_{q_1+1}^{q_2} f(p, q) + \sum_{q_1+1}^{p_1} \sum_{q_1+1}^{q_2} f(p, q) \right| \\
&\leq \sum_{p_1+1}^{p_2} \sum_{q_1+1}^{q_2} |f(p, q)| + \sum_{q_1+1}^{p_1} \sum_{q_1+1}^{q_2} |f(p, q)| \\
&= |S(p_2, q_2) - S(p_1, q_1)|.
\]

Hence \( \{s(p, q)\} \) is a convergent double sequence, and so \( \sum f(n, m) \) is a convergent double series. \( \square \)