1. Suppose that \{f_n\} and \{g_n\} are increasing sequences of functions on an interval \(I\), and put \(u_n = \max(f_n, g_n)\). Prove that if \(f_n \rightarrow f\) a.e. on \(I\), and \(g_n \rightarrow g\) a.e. on \(I\), then \(u_n \rightarrow \max(f, g)\) a.e. on \(I\).

2. Let \(I = [0, 1]\). Find a function \(f \in U(I)\) such that \(-f \in U(I)\).

3. Suppose that \(\{f_n\} \subset L(I)\) satisfies \((\forall n) f_n(x) \geq 0\) a.e. on \(I\), and \(f_n \rightarrow f\) a.e. on \(I\), and \((\exists A < \infty)(\forall n) \int_I f_n \leq A\). Prove that the limit function \(f\) belongs to \(L(I)\) and that \(\int_I f \leq A\). (This is called Fatou’s Lemma).

4. Find, with proof, all \(p \in \mathbb{R}\) for which the Lebesgue integral \(\int_0^\infty x^p \sin(x^2) \, dx\) exists.

5. Prove that the following Lebesgue integrals exist:

\[
\int_0^1 (x \log x)^2 \, dx, \quad \int_0^1 \log x \log(1 - x)^2 \, dx, \quad \int_0^1 \frac{\sqrt{1 - x}}{\log x} \, dx.
\]

6. For each of the Lebesgue integrals and intervals \(I\) below, determine with proof the set \(S\) of values \(s \in \mathbb{R}\) for which it must exist for every function \(f \in L(I)\). For each \(s\) not in \(S\), find a bounded continuous \(f\) for which the Lebesgue integral fails to exist.

\[
\int_0^1 f(x) \cos(2\pi sx) \, dx, \quad \int_0^\infty f(x) e^{sx} \, dx, \quad \int_0^\infty \frac{f(x)}{x^2 + s^2} \, dx.
\]

7. Suppose that \(f\) is continuous on \([0, 1]\), \(f(0) = 0\), and \(f'(0)\) exists. Prove that the Lebesgue integral \(\int_0^1 f(x) x^{-3/2} \, dx\) exists.

8. Suppose that \(f \in L([0, 1])\) and put \(g_n(x) \overset{\text{def}}{=} f(x) \sin(nx)\) for integers \(n\).

(a) Prove that \(g_n\) also belongs to \(L([0, 1])\).

(b) Prove that \(\int_0^1 g_n \rightarrow 0\) as \(|n| \rightarrow \infty\). (Hint: first prove the result for step functions.)

9. Suppose that \(I = \mathbb{R}\) and \(f \in L(I)\). Put \(f_y(x) \overset{\text{def}}{=} f(x - y)\) for \(y \in \mathbb{R}\). Prove that \(\int_I |f_y - f| \rightarrow 0\) as \(y \rightarrow 0\). (Hint hint: see previous hint.)

10. If \(f\) is Lebesgue-integrable on an open interval \(I\) and if \(f'\) exists a.e. on \(I\), prove that \(f'\) is measurable on \(I\).