1. Let \( \sim \) denote the relation on \( \mathbb{R} \) defined by
\[ x \sim y \iff x - y \in \mathbb{Q} \]
Prove that \( \sim \) is an equivalence relation, namely:
(i) \( (\forall x \in \mathbb{R}) x \sim x \)
(ii) \( (\forall x, y \in \mathbb{R}) x \sim y \iff y \sim x \)
(iii) \( (\forall x, y, z \in \mathbb{R}) (x \sim y \text{ and } y \sim z) \Rightarrow x \sim z \)

2. Given \( x \in \mathbb{R} \), define the equivalence class \([x] = \{ y \in \mathbb{R} : x \sim y \}\), where \( \sim \) is the equivalence relation of exercise 1 above.
(a) Prove that \([x]\) is countably infinite for every \( x \in \mathbb{R} \).
(b) Prove that the number of distinct equivalent classes is uncountable.

3. Suppose that \( \{f_n\} \subset L(I) \) satisfies \( (\forall n) f_n(x) \geq 0 \) a.e. on \( I \), and \( f_n \rightarrow f \) a.e. on \( I \), and \( (\exists A < \infty)(\forall n) \int_I f_n(x) \leq A \). Prove that the limit function \( f \) belongs to \( L(I) \) and that \( \int_I f \leq A \). (this is called Fatou’s Lemma).

4. Find, with proof, all \( p \in \mathbb{R} \) for which the Lebesgue integral \( \int_0^\infty x^p \sin(x^2) \, dx \) exists.

5. Prove that the following Lebesgue integrals exist:
\[
\int_0^1 (x \log x)^2 \, dx, \quad \int_0^1 \log x \log(1-x)^2 \, dx, \quad \int_0^1 \frac{\sqrt{1-x}}{\log x} \, dx.
\]

6. For each of the Lebesgue integrals and intervals \( I \) below, determine with proof the set \( S \) of values \( s \in \mathbb{R} \) for which it must exist for every function \( f \in L(I) \). For each \( s \) not in \( S \), find a bounded continuous \( f \) for which the Lebesgue integral fails to exist.
\[
\int_0^1 f(x) \cos(2\pi sx) \, dx, \quad \int_0^\infty f(x)e^{sx} \, dx, \quad \int_0^\infty \frac{f(x)}{x^2 + s^2} \, dx.
\]

7. Suppose that \( f \) is continuous on \([0,1]\), \( f(0) = 0 \), and \( f'(0) \) exists. Prove that the Lebesgue integral \( \int_0^1 f(x)x^{-3/2} \, dx \) exists.
8. Suppose that $f \in L([0,1])$ and put $g_n(x) \overset{\text{def}}{=} f(x) \sin(nx)$ for integers $n$.
   (a) Prove that $g_n$ also belongs to $L([0,1])$.
   (b) Prove that $\int_0^1 g_n \to 0$ as $|n| \to \infty$. (Hint: first prove the result for step functions.)

9. Put $I = [0,1]$. Suppose that $f$ is continuous on $I$ with $f(0) = 0$, and that $f'(0)$ exists and is finite.
   (Here we mean the one-sided derivative
   \[ f'(0) \overset{\text{def}}{=} \lim_{h \to 0^+} \frac{f(h) - f(0)}{h}. \]
   Prove that $g(x) \overset{\text{def}}{=} f(x)/x^{3/2}$ belongs to $L(I)$. 

10. If $f$ is Lebesgue-integrable on an open interval $I$ and if $f'$ exists a.e. on $I$, prove that $f'$ is measurable on $I$. 
