1. Find a counterexample to show that the union of two σ-algebras for a set $X$ need not be a σ-algebra on $X$. (Hint: it is enough to consider a three-point set $X$.)

2. Show that the Borel sets $\mathcal{B}(\mathbb{R})$ are generated by the compact subsets of $\mathbb{R}$.

3. Let $(X, A)$ be a measurable space and suppose $\mu : A \to [0, +\infty]$ is a countably additive function on the σ-algebra $A$.
   (a) Show that if $\mu$ satisfies $\mu(A) < \infty$ for some $A \in A$, then $\mu(\emptyset) = 0$. (This implies that $\mu$ is a measure.)
   (b) Find an example $\mu$ for which $\mu(\emptyset) \neq 0$. (Thus the first property of a measure does not follow from countable additivity and non-negativity.)

4. Let $(X, A)$ be a measurable space. Say that a sequence of measures $\{\mu_n : n = 1, 2, \ldots\}$ is increasing iff
   \[
   (\forall A \in A)(\forall n) \mu_{n+1}(A) \geq \mu_n(A)
   \]
   (a) Show that if $\{\mu_n\}$ is an increasing sequence of measures, then
   \[
   \mu(A) \overset{\text{def}}{=} \lim_{n \to \infty} \mu_n(A)
   \]
   defines a measure on $(X, A)$.
   (b) Let $\{\mu_n : n = 1, 2, \ldots\}$ be any sequence of measures on $(X, A)$. Prove that
   \[
   \mu(A) \overset{\text{def}}{=} \sum_{n=1}^{\infty} \mu_n(A)
   \]
   defines a measure on $(X, A)$.

5. Let $(X, A)$ be a measurable space and define the function $\delta_x : A \to [0, +\infty]$ by
   \[
   \delta_x(A) = \begin{cases} 1, & \text{if } x \in A, \\ 0, & \text{otherwise.} \end{cases}
   \]
   Prove that $\delta_x$ is a measure.
6. Let \( \{x_n : n = 1, 2, \ldots\} \subset \mathbb{R} \) be a sequence of points and define a measure \( \mu \) on \((\mathbb{R}, \mathcal{B}(\mathbb{R}))\) by

\[
\mu = \sum_n \delta_{x_n},
\]

where \( \delta_x \) is the measure defined in exercise 5.

(a) Show that \( \mu \) assigns finite values to bounded intervals if and only if \( |x_n| \to +\infty \) as \( n \to \infty \).

(b) For what sequences is this \( \mu \) a \( \sigma \)-finite measure?

7. Show that a subset \( B \subset \mathbb{R} \) is Lebesgue measurable if and only if

\[
\lambda^*(I) = \lambda^*(I \cap B) + \lambda^*(I \cap B^c)
\]

for every open interval \( I \subset \mathbb{R} \).

8. Let \((X, \mathcal{A}, \mu)\) be a measure space and let \( \mu^* \) denote the outer measure defined by \( \mu \) using \( \mathcal{A} \), namely

\[
\mu^*(S) = \inf \{ \mu(A) : S \subset A, A \in \mathcal{A} \}
\]

Define the inner measure \( \mu_* : 2^X \to [0, +\infty] \) by

\[
\mu_*(S) = \sup \{ \mu(A) : A \subset S, A \in \mathcal{A} \}
\]

(a) Prove that \( \mu_*(S) \leq \mu^*(S) \) for every \( S \subset X \).

(b) Prove that for any subset \( S \subset X \) (not necessarily in \( \mathcal{A} \)) there are sets \( A_0, A_1 \in \mathcal{A} \) satisfying \( A_0 \subset S \subset A_1 \) and

\[
\mu(A_0) = \mu_*(S); \quad \mu^*(S) = \mu(A_1).
\]

9. Say that the measure space \((X, \mathcal{A}, \mu)\) is complete iff whenever \( S \subset X \) and there is some \( A \in \mathcal{A} \) with \( S \subset A \) and \( \mu(A) = 0 \), we may conclude that \( S \in \mathcal{A} \).

(a) Prove that if \( \mu^* \) is an outer measure on \( X \), and \( \mathcal{A} \) is the \( \sigma \)-algebra of \( \mu^* \)-measurable sets, then \((X, \mathcal{A}, \mu^*)\) is complete.

(b) Decide whether \((\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda)\) is complete.

10. Define the completion of the \( \sigma \)-algebra \( \mathcal{A} \) under \( \mu \) to be the collection \( \mathcal{A}_\mu \) of subsets \( S \subset X \) for which there exist \( E, F \in \mathcal{A} \) satisfying \( E \subset S \subset F \) and \( \mu(F \setminus E) = 0 \).

(a) Prove that \( \mathcal{A} \subset \mathcal{A}_\mu \).

(b) Prove that \( S \in \mathcal{A}_\mu \) if and only if \( \mu_*(S) = \mu^*(S) \). (Hint: assume exercise 8.)

(c) Prove that \( \mathcal{A}_\mu \) is a \( \sigma \)-algebra.