1. Verify that the trigonometric system \( \left\{ \frac{1}{\sqrt{\pi}} \sin nx : n = 1, 2, \ldots \right\} \cup \left\{ \frac{1}{\sqrt{\pi}} \cos nx : n = 1, 2, \ldots \right\} \cup \left\{ \frac{1}{\sqrt{2\pi}} \right\} \) is orthonormal in \( L^2([0, 2\pi]) \).

2. Let \( \{\phi_0, \phi_1, \ldots\} \subset X \) be an orthonormal system of functions, where \( X \) is an inner product space in which \( \|f\| = 0 \iff f = 0 \). Prove that the following three statements are equivalent:
   
   (a) If \( \langle f, \phi_n \rangle = \langle g, \phi_n \rangle \) for all \( n = 0, 1, 2, \ldots \), then \( f = g \).
   
   (b) If \( \langle f, \phi_n \rangle = 0 \) for all \( n \), then \( f = 0 \).
   
   (c) If \( T \) is an orthonormal system such that \( \{\phi_0, \phi_1, \ldots\} \subset T \), then \( T = \{\phi_0, \phi_1, \ldots\} \).

For Problems 3 and 4, define \( C([0, 1]) \) to be the space of complex-valued continuous functions on the compact interval \([0, 1]\), with \( \langle f, g \rangle \overset{\text{def}}{=} \int_0^1 f(t)\bar{g}(t)\,dt \) for \( f, g \in C([0, 1]) \), and \( \|f\| \overset{\text{def}}{=} \sqrt{\langle f, f \rangle} \).

3. For \( f \in C([0, 1]) \), prove that \( \|f\| = 0 \iff f = 0 \).

4. Prove that the set \( \{e^{2\pi int} : n \in \mathbb{Z}\} \subset C([0, 1]) \) is an orthonormal system satisfying all three conditions of Problem 2.

5. Put \( I = [0, 1] \subset \mathbb{R} \). Suppose that \( \mathcal{H} = \{\psi_n : n = 0, 1, 2, \ldots\} \subset L^2(I) \) is the orthogonal system of Haar functions defined for \( n = 0 \) by \( \psi_0 = 1_I \), and for \( 0 < n = 2^j + k \) with \( 0 \leq k < 2^j \) is defined by

\[
\psi_n(x) = \begin{cases} 
1, & \text{if } \frac{k}{2^j} \leq x < \frac{k+1}{2^j}, \\
-1, & \text{if } \frac{k+1}{2^j} \leq x < \frac{k+2}{2^j}, \\
0, & \text{otherwise}.
\end{cases}
\]

Show that if \( f \in L^2(I) \) and \( \langle f, \psi_n \rangle = 0 \) for all \( n = 0, 1, 2, \ldots \), then \( f = 0 \) a.e. on \( I \).

6. Show that \( x = \pi - 2 \sum_{n=1}^{\infty} \frac{\sin nx}{n} \), if \( 0 < x < 2\pi \).
7. Show that \( \frac{x^2}{2} = \pi x + 2 \sum_{n=1}^{\infty} \frac{\cos nx}{n^2} - 2 \sum_{n=1}^{\infty} \frac{1}{n^2} \), if \( 0 \leq x \leq 2\pi \). Conclude that \( \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \).

(Hint: integrate the formula from Problem 6.)

For Problems 8 and 9, define a \( 2\pi \)-periodic function \( f \) as follows:

\[
    f(t) = \begin{cases} 
        1, & \text{if } 0 < t < \pi; \\
        -1, & \text{if } -\pi < t < 0; \\
        0, & \text{if } t = -\pi, t = 0, \text{ or } t = \pi.
    \end{cases}
\]

8. Show that \( f(x) = \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\sin(2n-1)x}{2n-1} \) for every \( x \in \mathbb{R} \).

9. Let \( s_n(x) \) be the partial sum of the first \( n \) terms of the Fourier series of the function \( f \) defined above. Show that for any \( \epsilon > 0 \),

\[
    \lim_{n \to \infty} \max_{|x| < \epsilon} \left| s_n(x) - \min_{|x| < \epsilon} s_n(x) \right| = \frac{4}{\pi} \int_{0}^{\pi} \frac{\sin t}{t} \, dt.
\]

(Hint: see problem 11.19 on pp.338–339 of the text.)

This result is known as Gibbs’ phenomenon.

10. Prove that if \( f \in L([0, 2\pi]) \) and \( f'(x_0) \) exists at some point \( x_0 \in (0, 2\pi) \), then the Fourier series generated by \( f \) converges at \( x_0 \).