1. Classify the following singularities as removable, poles, or essential. If the singularity is a pole, state its order.

(a) \( \frac{1}{e^{z^2} - 1} \) at \( z = 0 \)

(b) \( e^{1/z} \) at \( z = 0 \)

(c) \( \frac{z}{\sin z} \) at \( z = 0 \)

**Solution:**

(a) Pole of order 2, since \( e^{z^2} - 1 = z^2 + O(z^4) \) as \( z \to 0 \).

(b) Essential singularity, since \( e^{1/zn} = 1 \) for \( zn = 1/(2n + 1) \pi i \to 0 \) but \( e^{1/zn} = -1 \) for \( zn = 1/(2n + 1) \pi i \to 0 \) as \( n \to \infty \). Hence the function can have no limit as \( z \to 0 \).

(c) Removable singularity, since \( \lim_{z \to 0} \frac{z}{\sin z} = 1/\lim_{z \to 0} \frac{(\sin z)/z}{1} = 1 \).

2. Find the residues of the following functions at the indicated points.

(a) \( \frac{1}{e^{z^2} - 1} \) at \( z = 0 \)

(b) \( \frac{z^4}{(z - \frac{1}{6}z^3 - \sin z)} \) at \( z = 0 \)

(c) \( \frac{(z^2 + 1)/z^4 - 1}{1} \) at \( z = 1 \) and \( z = i \).

**Solution:**

(a) Since this is a simple pole, find the residue as follows:

\[
\lim_{z \to 0} \frac{(z - 0)}{e^{z^2} - 1} = \lim_{z \to 0} \frac{z}{z + O(z^2)} = \lim_{z \to 0} \frac{1}{1 + O(z)} = 1.
\]

(b) Note that \( z - \frac{1}{6}z^3 - \sin z = -\frac{1}{120}z^5 + O(z^7) \) at \( z = 0 \). Hence \( \frac{z^4}{(z - \frac{1}{6}z^3 - \sin z)} \) has a simple pole at \( z = 0 \). Find the residue as follows:

\[
\lim_{z \to 0} \frac{(z - 0)z^4}{z - \frac{1}{6}z^3 - \sin z} = \lim_{z \to 0} \frac{z^5}{z - 0 - \frac{1}{120}z^5 + O(z^7)} = \lim_{z \to 0} \frac{-120}{1 + O(z^2)} = -120.
\]

(c) Factor the numerator and denominator polynomials into

\[
\frac{(z - i)(z + i)}{(z - i)(z + i)(z - 1)(z + 1)} = 1
\]

Hence \( z = i \) (also \( z = -i \)) is a removable singularity, so the residue there is 0. However, \( z = 1 \) is a simple pole with residue \( \lim_{z \to 1} \frac{(z-1)}{(z-1)(z+1)} = \frac{1}{2} \).
3. Find the residue of the function \( f(z) = \frac{1}{\sinh^2 z} \) at \( z = 0 \).

**Solution:** Since \( \sinh z = O(z) \) as \( z \to 0 \), we see that \( z = 0 \) is a pole of order 2 for \( \frac{1}{\sinh^2 z} \). Find the residue as follows, setting \( n = 2 \) in the formula on page 73 of our text:

\[
\lim_{z \to z_0} \frac{1}{(n-1)!} \left( \frac{d}{dz} \right)^{(n-1)} \left[ f(z)(z - z_0)^n \right] = \lim_{z \to z_0} \frac{d}{dz} \left[ \frac{z^2}{\sinh^2 z} \right]
\]

\[
= \lim_{z \to z_0} \frac{2z \sinh z - 2z^2 \cosh z}{\sinh^3 z}
\]

\[
= 2 \lim_{z \to z_0} \left[ \frac{z}{\sinh z} \right] \left[ \frac{\sinh z - z \cosh z}{\sinh^2 z} \right]
\]

L'Hôpital's rule allows us to evaluate the factor limits:

\[
\lim_{z \to 0} \left[ \frac{z}{\sinh z} \right] = \lim_{z \to 0} \left[ \frac{1}{\cosh z} \right] = 1,
\]

and

\[
\lim_{z \to 0} \left[ \frac{\sinh z - z \cosh z}{\sinh^2 z} \right] = \lim_{z \to 0} \left[ \frac{z \sinh z}{2 \sinh z \cosh z} \right] = 0.
\]

Hence the residue is 0. \( \square \)