1. Find a measure space \((X, \mathcal{A}, \mu)\), a subspace \(Y\) of \(L^1(X, \mu)\), and a bounded linear functional \(f\) on \(Y\) with norm 1 such that \(f\) has two distinct extensions to \(L^1(X, \mu)\) and each of the extensions has norm equal to 1.

2. Show that if \(1 \leq p < \infty\), then \(L^p([0,1])\) is separable, namely that there is a countable dense subset.

3. Show that \(L^\infty([0,1])\) is not separable, namely that any dense subset must be uncountable.

4. For \(k \geq 1\) and functions \(f : [0,1] \to \mathbb{R}\) that are \(k\) times differentiable, define

\[
\|f\|_{C^k} \overset{\text{def}}{=} \|f\|_{\infty} + \|f'\|_{\infty} + \cdots + \|f^{(k)}\|_{\infty},
\]

where \(f^{(k)}\) is the \(k\)th derivative of \(f\). Let \(C^k([0,1])\) be the collection of \(k\) times continuously differentiable functions \(f\) with \(\|f\|_{C^k} < \infty\). Is \(C^k([0,1])\) complete with respect to the norm \(\| \cdot \|_{C^k}\)?

5. Fix \(\alpha \in (0,1)\). For a continuous function \(f : [0,1] \to \mathbb{R}\), define

\[
\|f\|_{C^\alpha} \overset{\text{def}}{=} \sup_{x \in [0,1]} |f(x)| + \frac{\sup_{x \neq y \in [0,1]} |f(x) - f(y)|}{|x - y|^{\alpha}}.
\]

Let \(C^\alpha([0,1])\) be the set of continuous functions \(f\) with \(\|f\|_{C^\alpha} < \infty\). Is \(C^\alpha([0,1])\) complete with respect to the norm \(\| \cdot \|_{C^\alpha}\)?
6. For positive integers \( n \), let
\[
A_n \overset{\text{def}}{=} \left\{ f \in L^1([0,1]) : \int_0^1 |f(x)|^2 \, dx \leq n \right\}.
\]
Show that each \( A_n \) is a closed subset of \( L^1([0,1]) \) with empty interior.

7. Suppose \( L \) is a linear functional on a normed linear space \( X \). Prove that \( L \) is a bounded linear functional if and only if the set \( Z \overset{\text{def}}{=} \{ x \in X : Lx = 0 \} \) is closed.

8. Suppose \( X \) and \( Y \) are Banach spaces and \( \mathcal{L} \) is the collection of bounded linear maps from \( X \) into \( Y \), with the usual operator norm:
\[
\|L\| \overset{\text{def}}{=} \sup_{\|x\| \leq 1} \|Lx\|_Y.
\]
Define \((L + M)x \overset{\text{def}}{=} Lx + Mx\) and \((cL)x = c(Lx)\) for \( L, M \in \mathcal{L}, x \in X \), and scalar \( c \).
Prove that \( \mathcal{L} \) is a Banach space.
NOTE: see Remark 18.10 on textbook p.178.

9. Set \( A \) in a normed linear space is called convex if
\[
\lambda x + (1 - \lambda)y \in A
\]
whenever \( x, y \in A \) and \( \lambda \in [0,1] \).

a. Prove that if \( A \) is convex, then the closure of \( A \) is convex.

b. Prove that the open unit ball in a normed linear space is convex. (The open unit ball is the set of \( x \) such that \( \|x\| < 1 \).)

10. The unit ball in a normed linear space \( V \) is called strictly convex if \( \|\lambda f + (1 - \lambda)g\| < 1 \) whenever \( \|f\| = \|g\| = 1, f \neq g \in V \), and \( \lambda \in (0,1) \).
Let \((X, A, \mu)\) be a measure space.

a. Prove that, if \( 1 < p < \infty \), then the unit ball in \( L^p(X, \mu) \) is strictly convex.

b. Prove that if \( X \) contains two or more points, then the unit balls in \( L^1(X, \mu) \) and \( L^\infty(X, \mu) \) are not strictly convex.

11. Let \( X \) be a metric space containing two or more points. Prove that the unit ball in \( C(X) \) is not strictly convex.

12. Let \( f_n \) be a sequence of continuous functions on \( \mathbb{R} \) that converge at every point. Prove that for every compact subset \( K \subset \mathbb{R} \) there exists a number \( M \) such that \( \sup_n |f_n| \) is bounded by \( M \) on that interval.
13. Suppose $\|\cdot\|_1$ and $\|\cdot\|_2$ are two norms on a vector space $X$ such that $\|x\|_1 \leq \|x\|_2$ for all $x \in X$, and suppose $X$ is complete with respect to both norms. Prove that there exists a positive constant $c$ such that

$$
\|x\|_2 \leq c \|x\|_1
$$

for all $x \in X$.

14. Suppose $X$ and $Y$ are Banach spaces.

a. Let $X \times Y$ be the set of ordered pairs $(x, y)$, $x \in X$, $y \in Y$, with componentwise addition and multiplication by scalars. Define

$$
\|(x, y)\|_{X \times Y} \overset{\text{def}}{=} \|x\|_X + \|y\|_Y.
$$

Prove that $X \times Y$ is a Banach space.

b. Let $L : X \rightarrow Y$ be a linear map such that if $x_n \rightarrow x$ in $X$ and $Lx_n \rightarrow y$ in $Y$, then $y = Lx$. Such a map is called a closed map. Let $G$ be the graph of $L$, defined by

$$
G \overset{\text{def}}{=} \{(x, y) \in X \times Y : y = Lx\}.
$$

Prove that $G$ is a closed subset of $X \times Y$, hence is complete.

c. Prove that the function $(x, Lx) \mapsto x$ is continuous, injective, linear, and surjective from $G$ onto $X$.

d. Prove the closed graph theorem: If $L$ is a closed linear map from one Banach space to another (and hence by part b has a closed graph), then $L$ is a continuous map.

15. Let $X$ be the space of continuously differentiable functions on $[0,1]$ with the supremum norm and let $Y = C([0,1])$. Define $D : X \rightarrow Y$ by $Df = f'$. Show that $D$ is a closed map but not a bounded one.

16. Let $A$ be the set of real-valued continuous functions on $[0,1]$ such that

$$
\int_0^{1/2} f(x) \, dx - \int_{1/2}^1 f(x) \, dx = 1.
$$

Prove that $A$ is a closed convex subset of $C([0,1])$, but there does not exist $f \in A$ such that $\|f\| = \inf_{g \in A} \|g\|$. 

17. Let $A_n$ be the subset of the real-valued continuous functions on $[0,1]$ given by

$$
A_n \overset{\text{def}}{=} \{ f : (\exists x \in [0,1])(\forall y \in [0,1]) |f(x) - f(y)| \leq n|x - y| \}.
$$

a. Prove that $A_n$ is nowhere dense in $C([0,1])$. 

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b. Prove that there exist functions \( f \in C([0,1]) \) which are nowhere differentiable on \([0,1]\), namely \( f'(x) \) does not exist at any point \( x \in [0,1] \).

18. Let \( X \) be a linear space and let \( E \subset X \) be a convex set with \( 0 \in E \). Define a non-negative function \( \rho : X \to \mathbb{R} \) by

\[
\rho(x) \overset{\text{def}}{=} \inf \{ t > 0 : t^{-1}x \in E \},
\]

with the convention that \( \rho(x) = \infty = \inf \emptyset \) if no \( t > 0 \) gives \( t^{-1}x \in E \). This called the Minkowski functional defined by \( E \).

a. Show that \( \rho \) is a sublinear functional, namely it satisfies \( \rho(0) = 0 \), \( \rho(x + y) \leq \rho(x) + \rho(y) \), and \( \rho(\lambda x) = \lambda \rho(x) \) for all \( x, y \in X \) and all \( \lambda > 0 \).

b. Suppose in addition that \( X \) is a normed linear space and \( E \) is an open convex set containing \( 0 \). Prove that the Minkowski functional defined by \( E \) is finite at every \( x \in X \) and that \( x \in E \) if and only if \( \rho(x) < 1 \).

19. Let \( X \) be a linear space and let \( \rho : X \to \mathbb{R} \) be a sublinear functional that is finite at every point. Prove that

\[
|\rho(x) - \rho(y)| \leq \max(\rho(y - x), \rho(x - y))
\]
for every \( x, y \in X \).

20. Let \( X \) be a normed linear space, let \( E \subset X \) be an open convex set containing \( 0 \), and let \( \rho : X \to \mathbb{R} \) be the Minkowski functional defined by \( E \). (See exercise 18 part a.) Prove that \( \rho \) is continuous on \( X \).

21. Let \( X \) be a linear space and let \( \rho : X \to \mathbb{R} \) be a sublinear functional. Suppose that \( M \) is a subspace of \( X \) and \( f : M \to \mathbb{R} \) is a linear functional dominated by \( \rho \), namely

\[
f(x) \leq \rho(x), \quad x \in M.
\]

Prove that there exists a linear functional \( F : X \to \mathbb{R} \) that satisfies \( F(x) = f(x) \) for all \( x \in M \) and \( F(x) \leq \rho(x) \) for all \( x \in X \).

NOTE: this implies the Hahn-Banach theorem, 18.5 on textbook p.173, in the special case \( \rho(x) \overset{\text{def}}{=} \|x\| \).

22. Let \( X \) be a Banach space and suppose \( x \) and \( y \) are distinct points in \( X \). Prove that there is a bounded linear functional \( f \) on \( X \) such that \( f(x) \neq f(y) \).

Note: it may thus be said that there are enough bounded linear functionals on \( X \) to separate points.

23. Let \( X \) be a Banach space, let \( A \subset X \) be an open convex set, and let \( B \subset X \) be a convex set disjoint from \( A \). Prove that there exists a bounded real-valued linear functional \( f \) and a constant \( s \in \mathbb{R} \) such that \( f(a) < s \leq f(b) \) for all \( a \in A \) and all \( b \in B \).
Hint: Consider the difference set $E = A - B + (a_0 - b_0)$ for fixed $a_0 \in A$, $b_0 \in B$, and apply exercises 18, 20, and 21.

24. Let $X$ be a normed linear space. For any convex $B \subset X$, say that a subset $F \subset B$ is a face of $B$ if, given $x, y \in B$ and $0 < \theta < 1$ with $\theta x + (1 - \theta)y \in F$, one may conclude that $x, y \in F$.

a. Suppose $f$ is a bounded linear functional on $X$ and $B \subset X$ is a convex subset such that $eta \overset{\text{def}}{=} \sup \{ f(x) : x \in B \}$ is finite. Define $$ F \overset{\text{def}}{=} \{ x \in B : f(x) = \beta \}. $$

Prove that $F$ is a face of $B$.

b. Suppose $B$ is a convex set, $F \subset B$ is a face of $B$, and $G \subset F$ is any subset. Prove that $G$ is a face of $F$ if and only if $G$ is a face of $B$.

25. Let $X$ be a linear space. For any convex $B \subset X$, say that $e \in B$ is an extreme point of $B$ iff

$$(\forall x, y \in B)(\forall \theta \in (0, 1)) \quad e = \theta x + (1 - \theta)y \Rightarrow e = x = y.$$ 

a. Suppose $B$ is an open convex set in a normed linear space $X$. Prove that $B$ has no extreme points.

b. Suppose $B$ is a compact convex set in a Banach space $X$. Prove that if $B$ is non-empty then $B$ contains an extreme point.

(Hint: Apply Zorn’s lemma to the collection of closed non-empty faces of $B$ partially ordered by $F_1 \leq F_2$ iff $F_2$ is a face of $F_1$. Show that any maximal element contains a single point of $B$, which is therefore an extreme point.)

26. Let $X$ be a linear space and $A \subset X$ any subset. Define the convex hull of $A$ to be $$ \text{ch} (A) \overset{\text{def}}{=} \{ \theta x + (1 - \theta)y : x, y \in A; 0 \leq \theta \leq 1 \}. $$

If $X$ is a normed linear space, define the closed convex hull of $A$ to be the closure of $\text{ch} (A)$, and denote it by $\overline{\text{ch}} (A)$.

a. Prove that if $A \subset B \subset X$, then $\overline{\text{ch}} (A) \subset \overline{\text{ch}} (B)$.

b. Prove that if $A$ is a closed convex set, then $A = \overline{\text{ch}} (A)$.

27. Let $X$ be a Banach space and suppose that $A \subset X$ is compact and convex. Let $E \subset A$ be the set of extreme points of $A$ as defined in exercise 25. Prove that $A = \overline{\text{ch}} (E)$.

Hint: use exercises 23 and 26.

Note: this is the Krein-Milman theorem.