Fraying may be performed over an arbitrary reach interval \([\alpha - \epsilon, \alpha + \epsilon]\), using the formula:

\[
F(r, \alpha, \epsilon)u(t) = \begin{cases} 
  r \left( \frac{t-\alpha}{\epsilon} \right) u(t) + r \left( \frac{\alpha - t}{\epsilon} \right) u(2\alpha - t), & \text{if } \alpha < t < \alpha + \epsilon, \\
  \tilde{r} \left( \frac{t-\alpha}{\epsilon} \right) u(t) - \tilde{r} \left( \frac{\alpha - t}{\epsilon} \right) u(2\alpha - t), & \text{if } \alpha - \epsilon < t < \alpha, \\
  u(t), & \text{otherwise.}
\end{cases}
\] (3.20)

The formula for \(S(r, \alpha, \epsilon)\), or splicing over \([\alpha - \epsilon, \alpha + \epsilon]\), is similar and left as an exercise. It is mostly shown in Equation 3.23 further on.

The boundary conditions at \(\alpha\) will be the same as the boundary conditions at zero described in Lemma 3.4. Likewise, splicing over this reach interval undoes the boundary conditions at \(\alpha\). Every \(\epsilon > 0\) will yield the same boundary conditions.

Suppose \(F_1 = F(r_1, \alpha_1, \epsilon_1)\) and \(F_2 = F(r_2, \alpha_2, \epsilon_2)\) are fraying operators with reach intervals \(B_1 = [\alpha_1 - \epsilon_1, \alpha_1 + \epsilon_1]\) and \(B_2 = [\alpha_2 - \epsilon_2, \alpha_2 + \epsilon_2]\), respectively. If \(B_1\) and \(B_2\) are disjoint, then \(F_1\) and \(F_2\) can be evaluated as follows:

\[
F_1F_2u(t) = \begin{cases} 
  F_1u(t), & \text{if } t \in B_1; \\
  F_2u(t), & \text{if } t \in B_2; \\
  u(t), & \text{otherwise.}
\end{cases}
\] (3.21)

The same formula may be used to evaluate \(F_2F_1u(t)\), so the operators \(F_1\) and \(F_2\) commute. Likewise, splicing operators \(S_1 = S(r_1, \alpha_1, \epsilon_1)\) and \(S_2 = S(r_2, \alpha_2, \epsilon_2)\) will commute with each other:

\[
S_1S_2v(t) = \begin{cases} 
  S_1v(t), & \text{if } t \in B_1; \\
  S_2v(t), & \text{if } t \in B_2; \\
  v(t), & \text{otherwise,}
\end{cases}
= S_2S_1v(t).
\] (3.22)

Similar formulas show that \(S_1\) commutes with \(F_2\) and \(S_2\) commutes with \(F_1\). The remaining pairs \(S_1, F_1\) and \(S_2, F_2\) commute because they are inverses.

Let \(\alpha < \beta\) define an interval \(I = [\alpha, \beta]\), and choose \(0 < \epsilon < \frac{1}{2}(\beta - \alpha)\). A smooth function \(u\) frayed at \(t = \alpha\) and \(t = \beta\) with reach intervals \(B_t(\alpha)\) and \(B_t(\beta)\), respectively, may have its ends spliced together with the loop operator:

\[
L(r, [\alpha, \beta], \epsilon)u(t) = \begin{cases} 
  \tilde{r} \left( \frac{t-\alpha}{\epsilon} \right) u(t) - \tilde{r} \left( \frac{\alpha - t}{\epsilon} \right) u(\alpha + \beta - t), & \text{if } \alpha < t \leq \alpha + \epsilon, \\
  r \left( \frac{t-\alpha}{\epsilon} \right) u(t) + r \left( \frac{\alpha - t}{\epsilon} \right) u(\alpha + \beta - t), & \text{if } \beta - \epsilon \leq t < \beta, \\
  u(t), & \text{otherwise;}
\end{cases}
\]

\[
= \begin{cases} 
  S(r, \alpha, \epsilon)u_I(t), & \text{if } \alpha < t \leq \alpha + \epsilon, \\
  S(r, \beta, \epsilon)u_I(t), & \text{if } \beta - \epsilon \leq t < \beta,
\end{cases}
\] (3.23)

Here \(u_I\) is the periodic extension of \(u\) from its localization to \(I = [\alpha, \beta]\), as defined in Equation 3.7.

The smooth local periodization of a function is a combination of fraying at two points and splicing into a loop. Namely, suppose \(u = u(t)\) is a smooth function and \(I = [\alpha, \beta]\) is an interval. Choose a smooth rising cut-off function \(r\) and a positive