Finally, observe that

\[ r \left( \frac{t - \alpha}{\epsilon} \right) = \begin{cases} 1, & \text{if } t \geq \alpha + \epsilon, \\ 0, & \text{if } t \leq \alpha - \epsilon, \end{cases} \quad r \left( \frac{\alpha - t}{\epsilon} \right) = \begin{cases} 0, & \text{if } t \geq \alpha + \epsilon, \\ 1, & \text{if } t \leq \alpha - \epsilon. \end{cases} \]

These valuations result in the formula in Equation 3.20.

d. Performing the previous calculation, but starting from Equation 3.15, we get

\[
S(r, \alpha, \epsilon)u(t) = \tau_\alpha \delta_\epsilon S \delta_\epsilon^{-1} \tau_\alpha^{-1} u(t) = \begin{cases} r(\frac{\alpha}{\epsilon})u(t) - r(\frac{\alpha-\epsilon}{\epsilon})u(2\alpha - t), & \text{if } \alpha < t \leq \alpha + \epsilon, \\ r(\frac{\alpha+\epsilon}{\epsilon})u(t) + r(\frac{\alpha}{\epsilon})u(2\alpha - t), & \text{if } \alpha - \epsilon \leq t < \alpha, \\ u(t), & \text{otherwise.} \end{cases}
\]

Finally, since the inverse of \( \tau_\alpha \delta_\epsilon \) is \( \delta_\epsilon^{-1} \tau_\alpha^{-1} = \delta_1/\tau_\alpha \), we can simplify

\[
S(r, \alpha, \epsilon)F(r, \alpha, \epsilon) = (\tau_\alpha \delta_\epsilon S \delta_\epsilon^{-1} \tau_\alpha^{-1}) (\tau_\alpha \delta_\epsilon F \delta_\epsilon^{-1} \tau_\alpha^{-1}) = \tau_\alpha \delta_\epsilon SF \delta_\epsilon^{-1} \tau_\alpha^{-1} = \tau_\alpha \delta_\epsilon \delta_\epsilon^{-1} \tau_\alpha^{-1} = Id,
\]

since \( SF = Id \). The other order, \( F(r, \alpha, \epsilon)S(r, \alpha, \epsilon) = Id \), likewise follows from \( FS = Id \). \( \square \)

3. **Solution:** Suppose that \( r \) is a rising cut-off function and \( \epsilon < T/2 \). Write \( F_0 = F(r, 0, \epsilon) \) and \( F_T = F(r, T, \epsilon) \) for the fraying operators at 0 and T, respectively. Since \( u(t+T) = u(t) \) for all \( t \), the function \( v \overset{\text{def}}{=} F_0 F_T u \) satisfies \( v(t+T) = v(t) \) for all \( -\frac{T}{2} \leq t \leq \frac{T}{2} \). In particular, that means \( v(t) = v(t) \) for all \( -\epsilon \leq t \leq T + \epsilon \). Also, \( v(t) = u(t) \) if \( t \) lies between \( \epsilon \) and \( T - \epsilon \), outside the reach intervals of \( F_0 \) and \( F_T \).

Now put \( I = [0, T] \). The second formula for the loop operator, Equation 3.23, simplifies to

\[
L_I v(t) = \begin{cases} S(r, 0, \epsilon) v(t), & \text{if } 0 < t \leq \epsilon; \\ S(r, T, \epsilon) v(t), & \text{if } T - \epsilon \leq t < T; \\ v(t), & \text{otherwise.} \end{cases}
\]

Writing \( S_0 = S(r, 0, \epsilon) \) and \( S_T = S(r, T, \epsilon) \) for the splicing operators at 0 and \( T \), one calculates

\[
L_I v(t) = \begin{cases} S_0 F_0 u(t) = u(t), & \text{if } 0 < t \leq \epsilon, \\ S_T F_T u(t) = u(t), & \text{if } T - \epsilon \leq t < T, \\ v(t) = u(t), & \text{if } \epsilon \leq t \leq T - \epsilon. \end{cases}
\]

Thus \( L_I F_0 F_T u = u \) on \( I \), so \( (L_I F_0 F_T)_u = u \) on \( R \). \( \square \)

4. **Solution:** (i) \( t > 1 \) implies \( F u(t) = \bar{u}(t) = u(t) \) and \( \bar{S} u(t) = 1 u(t) = u(t) \). Likewise, \( t < -1 \) implies \( F u(t) = 1 u(t) = u(t) \) and \( \bar{S} u(t) = \bar{u}(t) = u(t) \).