3.1. FOURIER ANALYSIS

Figure 3.1: Cosine-based window functions with 1 and 2 continuous derivatives

In other words, localization to $I$ multiplies $f$ by $1_I$, the indicator function of the interval. The definition may also be applied to other kinds of intervals, like $I = (a, b)$ or $I = [a, b]$.

In general, a function $f = f(t)$ is said to have compact support if there is some closed and bounded interval $I = [a, b]$ such that $f(t) = 0$ for all $t \notin I$. Such an $f$ is said to be supported on $I$. A simple example is the zero function, which has compact support using any $I$. Less special is $1$, which has compact support and is supported on $[0, 1]$. The function $g(t) = e^{-t^2}$, which is never zero and therefore not compactly supported, may be described as supported on all of $\mathbb{R}$. But the restriction of any function $f$ to a bounded interval $I$ will have compact support, and $1_I f$ will be supported in $I$.

Even if $f$ is continuous, its localization need not be: there may be jump discontinuities at the endpoints $a$ and $b$ of $I$. But if $f(a) = f(a+) = 0$ and $f(b) = f(b-) = 0$ and $f$ is continuous at each $t \in (a, b)$, then $1_I f$ is continuous. If in addition $f$ satisfies a Lipschitz condition on $I$, then the localization $1_I f$ will satisfy a Lipschitz condition on all of $\mathbb{R}$.

A more sophisticated solution is to smooth the cut by pinching the ends down to zero. Let $u = u(t)$ be a “window function” such as the one defined below, which is called a Hanning window:

$$u(t) = \begin{cases} 0, & \text{if } t < 0 \text{ or } t > 1; \\ \frac{1}{2} - \frac{1}{2} \cos 2\pi t, & \text{if } 0 \leq t \leq 1. \end{cases} \tag{3.5}$$

its graph is plotted in the left half of Figure 3.1. This function $u$ has one continuous derivative: $u'(0+) = 0 + \pi \sin(2\pi 0+) = 0 = u'(0-)$ and $u'(1-) = 0 + \pi \sin(2\pi -) = 0 = u'(1+)$, so the derivatives of the pieces match up at $t = 0$ and $t = 1$.

The Hanning window is one member of a family of functions of the form

$$u(t) = \begin{cases} A - B \cos 2\pi t + C \cos 4\pi t, & \text{if } t \in [0, 1], \\ 0, & \text{otherwise}. \end{cases}$$

Such a $u$ can be made smoother by choosing $C \neq 0$ as follows: note that for all windows in the family, $u(0+) = A - B + C = u(1-)$, $u'(0+) = u'(1-) = 0$, and

$$u''(0+) = 4\pi^2 B - 16\pi^2 C = u''(1-)$$.