A.5. ... TO CHAPTER 5 EXERCISES 253

Thus, it is legal to differentiate once or twice with respect to \( a \) or \( b \) under the integral sign:

\[
\frac{\partial}{\partial a} W_u(a, b) = \int_{-\infty}^{\infty} \xi e^{-2\pi i b \xi} \phi'(a \xi) F_u(\xi) \, d\xi;
\]

\[
\frac{\partial}{\partial b} W_u(a, b) = -2\pi i \int_{-\infty}^{\infty} \xi e^{-2\pi i b \xi} \phi(a \xi) F_u(\xi) \, d\xi.
\]

It remains to show that these derivatives are continuous functions of \( a, b \) away from the line \( a = 0 \). But in both cases, this follows from the observation that the integrands are continuous functions of \( a, b \).

8. **Solution:** Since \( w = F1_{[-\frac{1}{2}, \frac{1}{2}]} \), use Plancherel’s theorem to compute \( \|w\| = \|F1_{[-\frac{1}{2}, \frac{1}{2}]}\| = \|1_{[-\frac{1}{2}, \frac{1}{2}]}\| = 1 \).

9. **Solution:** By the previous solution and by combining integrals, calculate that \( Fw = 1_{[-1, -\frac{1}{2}]}[\frac{1}{2}, 1] \). Thus,

\[
c_w = \int_{0}^{\infty} \frac{|1_{[-1, -\frac{1}{2}]}[\frac{1}{2}, 1](\xi)|^2}{\xi} \, d\xi = \int_{\frac{1}{2}}^{1} \frac{d\xi}{\xi} = \log 2 \approx 0.69315 < \infty.
\]

But \( F(-\xi) = F(\xi) \), so the \(-\xi\) integral is the same, so \( w \) is admissible.

10. **Solution:** The Fourier integral transform of \( w \) is

\[
\int_{-\infty}^{\infty} e^{-2\pi i x \xi} w(x) \, dx.
\]

Since \( w(x) = 1 \) if \( 0 < x < \frac{1}{2} \) and \( w(x) = -1 \) if \( \frac{1}{2} < x < 1 \), that simplifies to

\[
\int_{0}^{\frac{1}{2}} e^{-2\pi i x \xi} - \int_{\frac{1}{2}}^{1} e^{-2\pi i x \xi} = \frac{(e^{-\pi i \xi} - 1)^2}{2\pi i \xi}.
\]

11. **Solution:** It is necessary to show that \( \langle \phi_j, \phi_k \rangle = \delta(j - k) \). But Plancherel’s theorem allows writing

\[
\langle \phi_j, \phi_k \rangle = \langle F\phi_j, F\phi_k \rangle = \int_{-1/2}^{1/2} e^{2\pi i (k-j) \xi} \, d\xi = \delta(j - k),
\]

since \( F\phi_k(\xi) = e^{2\pi i k \xi} \text{sinc}(\xi) = e^{2\pi i k \xi} 1_{[-\frac{1}{2}, \frac{1}{2}]}(\xi) \).

12. **Solution:** Show that \( \sum_k g(2k) = -\sum_k g(2k + 1) = \frac{1}{\sqrt{2}} \):

\[
\sum_k g(2k) = \sum_k (-1)^{2k} h(2M - 1 - 2k) = \sum_k h(2M - 1 - 2k)
\]

\[
= \sum_k h(2(M - k) - 1) = \sum_i h(2i + 1) = \frac{1}{\sqrt{2}}.
\]