2. If $A$ is a subset of the set $S$, show that

(a) $(A^c)^c = A$;
(b) $A \cup A = A \cap A = A \cup \emptyset = A$;
(c) $A \cap \emptyset = \emptyset$;
(d) $A \times \emptyset = \emptyset$.

Solution:
(a) Let $x \in (A^c)^c$. Then $x \notin A^c$, which is the same as $x \in A$, and so $(A^c)^c \subset A$. If $x \in A$, then $x \notin A^c$, and so $x \in (A^c)^c$, which gives $A \subset (A^c)^c$. These two containments prove that $A = (A^c)^c$.
(b) We clearly have that $A \subset A \cup A$. Suppose that $a \in A \cup A$, then $a \in A$ or $a \in A$, which, in either case gives that $a \in A$ and $A \cup A \subset A$, and so $A \cup A = A$. We also clearly have that $A \cap A \subset A$. Let $a \in A$, then we have that $a \in A$ and $a \in A$, so $a \in A \cap A$, and so $A \subset A \cap A$. This proves $A \cap A = A$. $A = A \cup \emptyset$ is similar to (c), so we omit it from the solutions.
(c) We have that $\emptyset \subset A \cap \emptyset$ by definition. Let $a \in A \cap \emptyset$, then $a \in A$ and $a \in \emptyset$. This gives that $A \cap \emptyset \subset \emptyset$.
(d) This is similar to (c) as well, and we omit the solution.

3. Let $A, B, C$ be subsets of $S$. Prove the following statements

(a) $A^c \cup B^c = (A \cap B)^c$;
(b) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$;
(c) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$;

Solution:
(a) Let $x \in (A \cap B)^c$ and so $x \notin A \cap B$. This means that either $x \notin A$ and $x \notin B$ or $x \notin A$ and $x \in B$. If $x \in A$ and $x \notin B$, then $x \in A$ and $x \in B^c$. So we have that $x \in A^c \cup B^c$. Similarly, if $x \in A$ and $x \in B$, then we have $x \in A^c \cup B^c$. This gives that $(A \cap B)^c \subset A^c \cup B^c$.

Suppose that $x \in A^c \cup B^c$, then $x \in A^c$ or $x \in B^c$. Consider the situation where $x \in A^c$. If $x \in B$ then $x \notin A \cap B$, or equivalently $x \notin (A \cap B)^c$. If $x \in B^c$, then we have that $x \notin A \cap B$ as well and so $x \in (A \cap B)^c$, giving that $A^c \cup B^c \subset (A \cap B)^c$. The case when $x \in B^c$ is handle similarly.

(b) Let $x \in A \cap (B \cup C)$. Then $x \in A$ and $x \in B \cup C$. So we have that $x \in A$ and $x \in B$ or $x \in C$. If $x \in B$, then we have that $x \in A \cap B$, and so $A \cap (B \cup C) \subset (A \cap B) \subset (A \cap C)$. 

Homework 1

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7. Let \( f: X \rightarrow Y \) be a function and let \( A \) and \( B \) be subsets of \( X \) and \( C \) and \( D \) subsets of \( Y \). Prove that

(a) \( f(A \cup B) = f(A) \cup f(B) \);

(b) \( f(A \cap B) \subseteq f(A) \cap f(B) \);

(c) \( f^{-1}(C \cup D) = f^{-1}(C) \cup f^{-1}(D) \);

(d) \( f^{-1}(C \cap D) = f^{-1}(C) \cap f^{-1}(D) \);

(e) \( f^{-1}(f(A)) \supseteq A \);

(f) \( f(f^{-1}(C)) \supseteq C \).
1. For each $k \in \mathbb{N}$ let $A_k$ be a countable set. Show that

$$
\bigcup_{k=1}^{\infty} A_k
$$

Under the assumptions of Problem 7, prove that $f$ is one to one if and only if $\supset$ in (e) can be replaced by $=$ for all $A \subset X$, and $f$ is onto if and only if the sign $\supset$ in (f) can be replaced by $=$ for all $C \subset Y$.

**Solution:** Suppose that we have $f^{-1}(f(A)) = A$ for all $A \subset X$. We need to show that $f$ is one-to-one. Let $A = \{a\}$, a set with just one element. Then we have that $\{a\} = f^{-1}(\{f(a)\})$. If $f$ is not one-to-one, then there exists $a_1 \neq a_2 \in A$ such that $f(a_1) = f(a_2)$. Note that we then have that $\{a_1\} \neq \{a_2\}$ and $\{f(a_1)\} = \{f(a_2)\}$. But we then have that

$$\{a_1\} = f^{-1}(\{f(a_1)\}) = f^{-1}(\{f(a_2)\}) = \{a_2\}$$

but this is a contradiction since $\{a_1\} \neq \{a_2\}$, and so $f$ is one-to-one.

Suppose now that $f$ is one-to-one. By 7(e) we have that

$$f^{-1}(f(A)) \supset A$$

and so we only need to show that $f^{-1}(f(A)) \subset A$. Let $x \in f^{-1}(f(A))$, then we have that $f(x) \in f(A)$. Since $f(x) \in f(A)$, there exists an $a \in A$ such that $f(x) = f(a)$. However, since $f$ is one-to-one, we have that $x = a$. Thus, we have that $x \in A$, proving that $f^{-1}(f(A)) \subset A$.

**Additional Problems:**

1. For each $k \in \mathbb{N}$ let $A_k$ be a countable set. Show that

$$
\bigcup_{k=1}^{\infty} A_k
$$
is countable. In other words, a countable union of countable sets is countable.

**Solution:** Since $A_k$ is countable, we have can arrange them in a sequence $a^1_k, a^2_k, a^3_k, \ldots$. Then we write these sequences in the following way:

\[
\begin{array}{cccc}
a^1_1 & a^2_1 & a^3_1 & a^4_1 \\
a^1_2 & a^2_2 & a^3_2 & a^4_2 \\
a^1_3 & a^2_3 & a^3_3 & a^4_3 \\
\vdots & \vdots & \vdots & \vdots \\
a^1_k & a^2_k & a^3_k & a^4_k \\
\vdots & \vdots & \vdots & \vdots 
\end{array}
\]

Define $\phi : \mathbb{N} \to \bigcup_{k=1}^{\infty} A_k$ by following the zig-zag path starting in the upper left corner and then filling out the diagonals perpendicular to the main diagonal. Namely, $\phi(1) = a^1_1$, $\phi(2) = a^2_1$, $\phi(3) = a^1_2$, $\phi(4) = a^2_2$, $\ldots$. It is clear that $\phi$ is a bijection.

2. Prove that the map $f : A \to B$ is a bijection if and only if there exists a map $g : B \to A$ such that $g \circ f = \text{Id}_A$ and $f \circ g = \text{Id}_B$. Show also that $g = f^{-1}$ and is uniquely determined.

**Solution:** If $f$ is a bijection, then we have that $g = f^{-1}$ and that $f^{-1} \circ f = \text{Id}_A$ and $f \circ f^{-1} = \text{Id}_B$. Suppose now that there exists a map $g : B \to A$ with the desired properties. First, observe that $f$ is onto. Let $b \in B$, and set $a = g(b)$. Then we have that $f(a) = f(g(b)) = b$. Next, we observe that $f$ is one-to-one. Suppose that $a_1 \neq a_2$, then we claim that $f(a_1) \neq f(a_2)$ as well. If it was the case that $a_1 \neq a_2$ and $f(a_1) = f(a_2)$, then we would have that $a_1 = g(f(a_1)) = g(f(a_2)) = a_2$, which would be a contradiction.

To see that $g$ is uniquely determined, suppose that there are two maps $g_1 : B \to A$ and $g_2 : B \to A$, then we claim that $g_1(b) = g_2(b)$ for all $b \in B$.

3. Let $f : A \to B$ and $g : B \to C$ be bijections. Then $g \circ f$ is a bijection and $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.

**Solution:** It is easy to see that for the function $h = f^{-1} \circ g^{-1} : C \to A$ we have satisfies $h \circ (g \circ f) = \text{Id}_A$ and $(g \circ f) \circ h = \text{Id}_C$ and so by Problem 2 above we have that $g \circ f$ is a bijection and that $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.

4. Let $\mathcal{A} = \{A_\alpha\}$ be a collection of subsets of a set $S$. Prove the following two statements:

(a) $S \setminus \bigcup_\alpha A_\alpha = \bigcap_\alpha S \setminus A_\alpha$;
(b) $S \setminus \bigcap_\alpha A_\alpha = \bigcup_\alpha S \setminus A_\alpha$.

**Solution:** Part (a) is simply 5(b) from above. To see that (b) holds, let $B_\alpha = A_\alpha^c$. By part (a), we have

$$
\left( \bigcup_{\alpha} B_\alpha \right)^c = \bigcap_{\alpha} (B_\alpha)^c
$$

However, we have that $B_\alpha^c = A_\alpha$, and the complement of the complement is the original set, so we have

$$
\bigcup_{\alpha} A_\alpha^c = \bigcup_{\alpha} B_\alpha = \left( \left( \bigcup_{\alpha} B_\alpha \right)^c \right)^c = \left( \bigcap_{\alpha} A_\alpha \right)^c
$$

which is (b).