Math 4318 : Real Analysis II
Mid-Term Exam 2
28 March 2013

Name: ____________________

Definitions:

True/False:

Proofs:
1. ______________________

2. ______________________

3. ______________________

4. ______________________

5. ______________________

6. ______________________

Total:
Definitions and Statements of Theorems

1. (3 points) For a sequence of real numbers \( \{a_n\} \), state the definition for the series \( \sum a_n \) to converge.

Solution: Let \( s_n = \sum_{k=1}^{n} a_k \). Then the series \( \sum a_k \) converges if and only if the sequence of partial sums \( \{s_k\} \) converges.

2. (4 points) State the Alternating Series Theorem.

Solution: If \( \{a_k\} \) is a sequence of decreasing numbers converging to 0, then

\[
\sum (-1)^n a_n
\]

converges.
3. (3 points) For a sequence of functions \( \{f_n\} \) defined on an interval \([a, b]\), state the definition for the sequence \( f_n \) to converge uniformly to a function \( f \) on the interval \([a, b]\).

**Solution:** The sequence \( f_n \) converges uniformly to the function \( f \) if for any \( \epsilon > 0 \) there exists an integer \( N = N(\epsilon) \) such that for all \( x \in [a, b] \) and for all \( n \geq N \)

\[
|f_n(x) - f(x)| < \epsilon.
\]
True or False (1 point each)

1. If the series $\sum a_n$ converges, then $\lim_{n \to \infty} a_n = 0$.

   **Solution:** True

2. If $0 \leq a_n \leq b_n$ for all $n$ and $\sum a_n$ diverges, then $\sum b_n$ diverges.

   **Solution:** True

3. If $0 \leq a_n \leq b_n$ for all $n$ and $\sum a_n$ converges, then $\sum b_n$ converges.

   **Solution:** False

4. For $1 < p < \infty$ the series $\sum \frac{1}{n \log n}^p$ converges.

   **Solution:** True

5. The radius of convergence of a power series $f(x) = \sum_{k=0}^{\infty} a_k x^k$ is given by

   $$\left( \limsup_{n \to \infty} |a_n|^\frac{1}{n} \right)^{-1}.$$ 

   **Solution:** True

6. If $R > 0$ then $f(x) = \sum_{k=0}^{\infty} \frac{x^k}{R^k}$ has radius of convergence $R$.

   **Solution:** True

7. If $f(x) = \sum_{k=0}^{\infty} a_k x^k$ has radius of convergence $R$, then $f(x)$ belongs to $C^\infty(-R, R)$. 

8. If $\sum a_n$ converges, then $\sum |a_n|$ converges too.

Solution: False

9. If $a_n \geq 0$ for all $n \in \mathbb{N}$, $\sum a_n$ converges and $p > 1$ then $\sum a_n^p$ converges.

Solution: True

10. If $\{a_n\}$ is a sequence of non-zero real numbers, then

$$\liminf_{n \to \infty} |a_n|^\frac{1}{n} \leq \liminf_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right|.$$

Solution: False
Proofs

1. (5 points) Suppose that for sequence of real numbers $\{a_n\}$ and $\{b_n\}$ that

$$0 \leq b_{n+1} \leq a_n \leq b_n \quad \forall n \in \mathbb{N}.$$ 

Prove that the series $\sum a_n$ converges if and only if the series $\sum b_n$ converges.

**Solution:** Use the comparison test twice.
2. (10 points) Calculate

\[
\lim_{n \to \infty} \int_0^1 \frac{nx}{1 + n^2x} \, dx
\]

**Solution:** If we have

\[
\lim_{n \to \infty} \frac{nx}{1 + n^2x} = 0
\]

uniformly for \( x \in [0, 1] \), then

\[
\lim_{n \to \infty} \int_0^1 \frac{nx}{1 + n^2x} \, dx = \int_0^1 \lim_{n \to \infty} \frac{nx}{1 + n^2x} \, dx = 0.
\]

So, we are left showing that

\[
\lim_{n \to \infty} \frac{nx}{1 + n^2x} = 0
\]

uniformly for \( x \in [0, 1] \). Clearly when \( x = 0 \) we have that the limit is 0, so we can focus on \( x \neq 0 \). But, then we have

\[
\frac{nx}{1 + n^2x} \leq \frac{nx}{n^2x} = \frac{1}{n}.
\]

Thus, given \( \epsilon > 0 \), choosing \( N = N(\epsilon) \) sufficiently large, \( N > \frac{1}{\epsilon} \) works, we have that

\[
\frac{nx}{1 + n^2x} < \epsilon
\]

for all \( x \in [0, 1] \).

An alternate proof is as follows:

\[
\frac{nx}{1 + n^2x} = \frac{1}{n} \frac{n^2x}{1 + n^2x} = \frac{1}{n} \left( \frac{1 + n^2x}{1 + n^2x} - \frac{1}{1 + n^2x} \right) = \frac{1}{n} - \frac{1}{n^2} \frac{1}{1 + n^2x}.
\]

This is then a simple integration problem from calculus, and we have

\[
\int_0^1 \frac{nx}{1 + n^2x} \, dx = \frac{1}{n} - \frac{1}{n^3} \ln(1 + n^2).
\]

Then it is enough to show that each term on the right goes to 0 as \( n \) approaches infinity. For the first term this is trivial, and for the second term this is an application of L’Hopital’s Rule.
3. (10 points) Let the radius of convergence of $\sum a_n x^n$ and $\sum b_n x^n$ be $R_1$ and $R_2$ respectively. Suppose that there exists a $N_0 \in \mathbb{N}$ such that for all $k \geq N_0$ we have that $|a_k| \leq |b_k|$. Prove that $R_2 \leq R_1$.

**Solution:** Recall that the radius of convergence of a power series $\sum c_n x^n$ is given by

$$R^{-1} = \limsup_{n \to \infty} |c_n|^{\frac{1}{n}}.$$ 

Let $k \geq N_0$, then we have that

$$\sup \left\{|a_n|^{\frac{1}{n}} : n > k \right\} \leq \sup \left\{|b_n|^{\frac{1}{n}} : n > k \right\}$$

since

$$|a_n| \leq |b_n| \quad n > k \geq N_0.$$ 

Thus, we have that

$$R_1^{-1} = \limsup |a_n|^{\frac{1}{n}} \leq \limsup |b_n|^{\frac{1}{n}} = R_2^{-1}.$$

Rearrangement gives that

$$R_2 \leq R_1.$$
4. (15 points) Let \( a, b \in \mathbb{R} \) with \( a < b \), and let \( \{f_k\} \) be a uniformly convergent sequence of continuous real-valued functions on \([a, b]\). Prove that

\[
\int_a^b \left( \lim_{n \to \infty} f_n(x) \right) \, dx = \lim_{n \to \infty} \int_a^b f_n(x) \, dx.
\]

Solution: This is the Theorem on page 138 from your text.
5. (20 points) Let \( f : \{ x \in \mathbb{R} : x \geq 1 \} \to \mathbb{R} \) be a decreasing positive-valued function. Prove that

\[
\sum_{n=1}^{\infty} f(n)
\]

converges if and only if

\[
\lim_{n \to \infty} \int_{1}^{n} f(x) \, dx
\]

exists.

**Solution:** This is Problem 9 on page 161 from your text.
6. (20 points)

(a) Show that the series
\[ \sum_{k=1}^{\infty} \frac{(-1)^k}{k + |x|} \]
converges for all \( x \in \mathbb{R} \).

(b) Recall that a function \( f(x) \) is Lipschitz if there exists a positive constant \( K \) so that
\[ |f(x) - f(y)| \leq K |x - y| \quad \forall x, y \in \mathbb{R}. \]

Show that the function \( g(x) := \sum_{k=1}^{\infty} \frac{(-1)^k}{k + |x|} \) is Lipschitz.

**Solution:** Fix \( x \in \mathbb{R} \). Set \( a_k(x) := \frac{1}{k + |x|} \). Then we have that \( a_k(x) \geq 0 \) for all \( k \in \mathbb{N} \), and since
\[ k + |x| \leq k + 1 + |x| \]
we have that \( a_{k+1}(x) \leq a_k(x) \). Also, note that \( \lim_k a_k(x) = 0 \) for all \( x \in \mathbb{R} \). Then by the Alternating Series Theorem, we have that
\[ \sum_{k=1}^{\infty} \frac{(-1)^k}{k + |x|} \]
must converge. Since \( x \in \mathbb{R} \) was arbitrary, we have convergence for all \( x \).

Set \( g(x) = \sum_{k=1}^{\infty} \frac{(-1)^k}{k + |x|} \). By part (a) we know this converges for all \( x \in \mathbb{R} \). Thus, we have
\[
g(x) - g(y) = \sum_{k=1}^{\infty} \frac{(-1)^k}{k + |x|} - \sum_{k=1}^{\infty} \frac{(-1)^k}{k + |y|} = \sum_{k=1}^{\infty} \frac{(-1)^k}{(k + |x|)(k + |y|)}
\]
Taking absolute values, and recalling that \( |x - |y|| \leq |x - y| \) and that \( k + |x| \geq k \) for all \( x \in \mathbb{R} \) we have
\[
|g(x) - g(y)| = ||y| - |x|| \sum_{k=1}^{\infty} \frac{(-1)^k}{(k + |x|)(k + |y|)} \\
\leq |x - y| \sum_{k=1}^{\infty} \frac{1}{(k + |x|)(k + |y|)} \\
\leq \left( \sum_{k=1}^{\infty} \frac{1}{k^2} \right) |x - y|.
\]
Set \( K = \sum_{k=1}^{\infty} \frac{1}{k^2} \), which is finite by the Integral test, and so we have
\[ |g(x) - g(y)| \leq K |x - y| \]
and \( g \) is Lipschitz.