Math/Music HW4 Solutions

March 23, 2005

Problem 1: Interval expressions

Solution:
(a) Up 67 cents: $\sqrt[\sqrt{2}]{2^{67}}$, 1.039 (b) Down 1050 cents: $\sqrt[\sqrt[3]{2}]{2^{-1050}}$, .545
(c) Up a major sixth: $\sqrt[\sqrt{2}]{2^{3}}$, 1.682
(d) The interval from $B_3$ to $G_1^+$: $\sqrt[\sqrt[3]{2}]{2^{-9}}$, .210

Problem 2: Frequencies

Solution:
(a) $C_4$: $440(2^{\frac{9}{12}}) \approx 261.626$ (b) $D_2^+$: $440(2^{\frac{-30}{12}}) \approx 77.782$
(c) $F_3$: $440(2^{\frac{16}{12}}) \approx 174.614$ (d) $E_3^+$: $440(2^{\frac{-42}{12}}) \approx 38.891$
(a) $A_4$: $256(2^{\frac{9}{12}}) \approx 430.539$ (b) $G_6^+$: $256(2^{\frac{30}{12}}) \approx 1448.155$
(b) $C_1$: $256(2^{\frac{36}{12}}) = 32$ (b) $F_2^+$: $256(2^{\frac{-18}{12}}) \approx 90.510$

Problem 3: Frequencies

Solution:
(a) $(E_3, G_3^+, B_3) = 220(2^{\frac{(-5, -1, 2)}{12}}) \approx (164.814, 207.652, 246.942)$
(b) $(F_4^+, A_4, G_6^+) = 440(2^{\frac{(-3, 0, 4)}{12}}) \approx (369.994, 440, 554.365)$
(c) $(A_5, C_6, E_6, G_6) = 880(2^{\frac{(0, 3, 7, 10)}{12}}) \approx (880, 1046.502, 1318.510, 1567.982)$
(d) $(A_3^+, B_3, D_4, F_4) = 880(2^{\frac{(-1, 2, 5, 8)}{12}}) \approx (207.652, 246.942, 293.665, 349.228)$

Problem 4: Interval equivalence modulo octave

Solution: All but the last are equivalent modulo octave.
(a) $5\frac{1}{20} = \frac{1}{4} = 2^{-2}$ (b) $14\frac{2}{7} = 4 = 2^2$ (c) $2.3\frac{1}{9/2} = \frac{1}{4} = 2^{-2}$ (d) $1.04\frac{1}{7/3} = 8 = 2^3$ (e) $2\frac{2}{3/5} = \frac{2}{3} \neq 2^n$

Problem 5: Frets

Solution: For each $n$ such that $1 \leq n \leq 12$, we have $r = 2^{\frac{n}{12}}$, $q = 1 - r^{-1} = 1 - 2^{\frac{n}{12}}$, and our fret placement $qL = 40(1 - 2^{\frac{n}{12}})$.

Problem 6: Period transformations

Solution: You can do this problem symbolically or in terms of graph transformations. Since most people did it (correctly) symbolically, I will do it mostly in terms of graph transformations. First of all, the shifts ($f(t) + c$ and $f(t - c)$), being translations, do not change the period or affect the tone. A vertical stretch ($c \cdot f(t)$) keeps the period the same but affects the amplitude by a factor of $c$. A horizontal stretch ($f(\frac{t}{c})$), however, does affect the period: stretching it out by $c$ multiplies the period by $c$. This adjusts the pitch by a factor of $c$ (lower if $c > 1$). We can also see this by plugging $t + cP$ to the function $f(\frac{t}{c})$. $f(\frac{t + cP}{c}) = f(\frac{t}{c} + \frac{cP}{c}) = f(\frac{t}{c} + P) = f(\frac{t}{c})$ since $f$ is periodic with period $P$. Therefore $f(\frac{t}{c})$ is periodic with period $cP$.

Problem 7: Tone transformations

Solution: Using (6) above, we see that

(a) $\frac{1}{2}f(t)$ has the same pitch but half the amplitude.

(b) $f(2t)$ has a frequency half that of $f$, so its tone is an octave lower.

(c) $f(t) + c$ is a vertical shift, so it does not affect the tone.

(d) $f(t + c)$ is a horizontal shift, so it does not affect the tone.

Problem 8: For the given tone, find $\alpha$ in $\sin(\alpha t)$.

Solution: We want to use the equation $\frac{\alpha}{2\pi} = \text{frequency}$, so $\alpha = 2\pi\text{frequency}$.

(a) $C_4$ has frequency 261.626 Hz, so $\alpha = 1643.842$.

(b) $A_2$ has frequency 103.83 Hz, so $\alpha = 652.359$.

(c) $D_6^\#$ has frequency 1244.508 Hz, so $\alpha = 7819.474$.

For Problems (9) and (10), the bad news was that was laboring under a misapprehension when grading (I had $A$ and $B$ mixed up). The good news is that I didn’t take any points off for that mistake (because nobody had it the same way as I did, funny how that works), so you may just have a whole lot of extra red that means nothing. However, there were some intricacies to the $\cos^{-1}$ and $\sin^{-1}$ functions that not everybody caught. I will, of course, be willing to talk over anything.
Problem 9: Sine transformations

Solution:

(a) We’re given the amplitude \( d = 5 \) and the phase shift \( \beta = \frac{\pi}{4} \). From \( k = \alpha = 30\pi \) we can find the period \( \frac{2\pi}{30\pi} = \frac{1}{15} \) and then the frequency is 15. Following the notes, \( a = \cos(\beta) = \cos(\frac{\pi}{4}) = \frac{\sqrt{2}}{2} \) and \( b = \sin(\beta) = \sin(\frac{\pi}{4}) = \frac{\sqrt{2}}{2} \). \( A = ad = \frac{5\sqrt{2}}{2} \) and \( B = bd = \frac{5\sqrt{2}}{2} \), so our function, then, is 
\[ f(t) = \frac{5\sqrt{2}}{2} \sin(30\pi t) + \frac{5\sqrt{2}}{2} \cos(30\pi t). \]

(b) We’re given the amplitude \( d = \sqrt{2} \) and the phase shift \( \beta = \pi \). From \( k = \alpha = 800 \) we can find the period \( \frac{2\pi}{800} = \frac{\pi}{400} \) and then the frequency is \( \frac{1000}{\pi} \). Following the notes, \( a = \cos(\beta) = \cos(\pi) = -1 \) and \( b = \sin(\beta) = \sin(\pi) = 0 \). \( A = ad = -\sqrt{2} \) and \( B = bd = 0 \), so our function, then, is 
\[ g(t) = -\sqrt{2} \sin(800t). \]

(c) We’re given the amplitude \( |d| = \frac{5}{3} \) and the phase shift \( \beta = \sin^{-1} .7 \). From \( k = \alpha = 2000 \) we can find the period \( \frac{2\pi}{2000} = \frac{\pi}{1000} \) and then the frequency is \( \frac{1000}{\pi} \). Following the notes, \( a = \cos(\beta) = \cos(\sin^{-1} .7) \approx .714 \) and \( b = \sin(\beta) = \sin(\sin^{-1} .7) = .7 \). \( A = ad \approx -1.19 \) and \( B = bd \approx -1.17 \), so our function, then, is 
\[ h(t) \approx -1.19 \cos(2000t) - 1.17 \sin(2000t). \]

Problem 10: Sine transformations

Solution:

(a) Here we’re given \( A = 4, B = 5, \) and \( k = \alpha = 300 \). We can find the period \( \frac{2\pi}{300} = \frac{\pi}{150} \) and the frequency \( \frac{150}{\pi} \) as before, while the amplitude is \( d = \sqrt{A^2 + B^2} = \sqrt{16 + 25} = \sqrt{41} \). \( \beta \), the phase shift, is the angle between the positive \( x \)-axis and the line connecting the point \( (A, B) \) to the origin, and since \( (A, B) \) is in the first quadrant we can find it by taking \( \cos^{-1} \frac{A}{d} = \cos^{-1} \frac{4}{\sqrt{41}} \approx .896 \). Our function, then, is 
\[ f(t) \approx \sqrt{41} \sin(300t + .896). \]

(b) Here we’re given \( A = 2, B = -2 \), and \( k = \alpha = 450\pi \). We can find the period \( \frac{2\pi}{450\pi} = \frac{\pi}{225} \) and the frequency 225 as before, while the amplitude is \( d = \sqrt{A^2 + B^2} = \sqrt{4 + 4} = 2\sqrt{2} \). \( \beta \), the phase shift, is the angle between the positive \( x \)-axis and the line connecting the point \( (A, B) \) to the origin, and since \( (A, B) \) is in the fourth quadrant we can find it by taking \( \sin^{-1} \frac{B}{d} = \sin^{-1} \frac{-2}{2\sqrt{2}} = -\frac{\pi}{4} \). Our function, then, is 
\[ g(t) = 2\sqrt{2} \sin(450\pi t - \frac{\pi}{4}). \]

(c) Here we’re given \( A = -1, B = 3 \), and \( k = \alpha = 1500\pi \). We can find the period \( \frac{2\pi}{1500\pi} = \frac{\pi}{750} \) and the frequency 750 as before, while the amplitude is \( d = \sqrt{A^2 + B^2} = \sqrt{1 + 9} = \sqrt{10} \). \( \beta \), the phase shift, is the angle between the positive \( x \)-axis and the line connecting the point \( (A, B) \) to the origin, and since \( (A, B) \) is in the second quadrant we can find it by taking \( \cos^{-1} \frac{A}{d} = \cos^{-1} \frac{-1}{\sqrt{10}} \approx 1.893 \). Our function, then, is 
\[ h(t) \approx \sqrt{10} \sin(1500\pi t + 1.893). \]