FINAL EXAM

Math 109 / Music 109A, Spring 2014

Name Solutions Id

Each problem is worth 10 points. Round off each decimal approximation to two digits to the right of the decimal.

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1. Give the (total) duration in beats of:

   (a) a quarter note in \( \frac{12}{8} \) time (compound time signature).

   \[ d = \frac{1}{4} \text{ beats} \]
   \[ \frac{12}{8} \text{ beats} = \frac{3}{2} \text{ beats} \]

   (b) a dotted eighth note in \( \frac{3}{2} \) time.

   \[ d = \frac{1}{4} \text{ beats} \]
   \[ \frac{3}{2} \text{ beats} = \frac{3}{4} \text{ beats} \]

   (c) a quarter note 7-tuplet in \( \frac{4}{4} \) time.

   \[ n + r = 2 \]
   \[ 4r = 2 \]

   \[ \frac{4}{4} \text{ beats} = 0 \text{ beats} \]

2. For the set \( \mathbb{Z} \) and a fixed positive integer \( m \), define the equivalence relation whose set of equivalence classes is \( \mathbb{Z}_m \). Show that it is in fact an equivalence relation and explain why there are exactly \( m \) equivalence classes. For \( m = 12 \) explain how this relates to keyboard intervals.

   So \( h \equiv l \) if \( m \mid (h - l) \).

   (i) \( m \mid (h - l) \) so \( h \equiv l \) (reflexive)

   (ii) If \( h \equiv l \), then \( m \mid (h - l) \) so \( h - l = a \cdot m \)

   \[ h - l = a \cdot m \]

   (iii) If \( h \equiv l \) and \( l \equiv r \), then \( m \mid (h - l) \) and \( m \mid (l - r) \)

   So \( h - l = a \cdot m \), \( l - r = b \cdot m \).

   So \( h - l + l - r = h - r = a \cdot m + b \cdot m = (a + b) \cdot m \), so \( m \mid (h - r) \) or \( h \equiv r \) (transitive)

   For \( n \in \mathbb{Z} \), \( n = zm + r \) with \( 0 \leq r < m \), so \( m \mid (h - r) \)

   So \( m \equiv r \) i.e., \( \{ n \mid m \} = \{ r \} \). So \( \{ 0 \}, \{ 1 \}, \ldots, \{ m - 1 \} \) are all the classes.

   If \( 0 \leq r \leq r' \leq m - 1 \), then \( m \mid (r' - r) \)

   So \( r' \neq r \), so \( \{ r \} \neq \{ r' \} \) and these classes are distinct. \( \mathbb{Z}_{12} \) gives the modular chromatic intervals.
3. Identify each chord in this major mode (Ionian) passage. Above the staff label each chord by root note class with suffix (e.g., E\textsuperscript{b7}). Below the staff, label each chord by root scale tone (e.g. bIII\textsuperscript{7}).

4. Convert to semitones the musical intervals given by the following ratios, indicating whether the interval is upward or downward.

(a) 0.9 \[ \left| 2 \log_2 \left( 0.9 \right) \right| \approx -1.32 \text{ (downward)} \]

(b) \[ \pi \left| 2 \log_2 \left( \pi \right) \right| \approx 19.82 \text{ (upward)} \]

Express as a ratio the following musical intervals.

(c) 137 cents \[ \frac{137}{1200} \approx 1.09 \]

(d) the just minor third \[ \frac{6}{5} = 1.20 \]

5. Add the needed sharps or flats to notes so that the following gives the Locrian scale tones 1 to 8, from C to C.
6. Suppose a 12-tone row chart begins: B, D, A, E, G, ... Write the upper left 5 \times 5 matrix of the resulting row chart. Then rewrite it replacing each note class with the element of \( \mathbb{Z}_{12} \) which measures its modular interval from B.

\[
\begin{array}{ccccc}
B & D & A & E & G \\
A & B & C & D & F \\
D & E & B & C & D \\
G & A & E & B & D \\
E & F & G & A & B \\
\end{array}
\]

\[
\begin{array}{cccccc}
\end{array}
\]

7. If the keyboard divided the octave into 9 rather than 12 equal intervals, what would be the generating intervals? Quote the relevant facts from modular arithmetic that justify your answer. Draw the circles that exhibit inverse pairs of these generating intervals.

We look for the generator of \( \mathbb{Z}_9 \). By theorem, \([n] \) is a generator iff \( \gcd(n, 9) = 1 \). Generators are: \([1], [2], [4], [5], [7], [8] \)
8. (a) Find the period, frequency, amplitude, and phase shift for the function

\[ g(t) = 4 \sin(880\pi t) + 3 \cos(880\pi t) \]

and express it in the form \( d \sin(\alpha t + \beta) \), giving a decimal approximation for \( \beta \). Identify the closest keyboard note to the pitch represented by this function, and the error (if any) in cents.

\[ d = \sqrt{4^2 + 3^2} = 5 \quad \text{(amplitude)} \]
\[ 790 = 2F \quad F = 395 \text{ Hz} = A_4 \quad \text{(error: 0 cents)} \]
\[ P = \frac{1}{F} = \frac{1}{440} \quad \text{(period)} \]
\[ g(t) = 5 \sin \left( 800 \pi t + \beta \right) \]
where \( \beta \approx 0.64 \quad \text{(phase shift)} \]
\[ = \arcsin \left( \frac{3}{5} \right) \]

(b) Find the period, frequency, amplitude, and phase shift for the function

\[ h(t) = 3\sqrt{2} \sin \left( 330\pi t + \frac{\pi}{4} \right) \]

and express it in the form \( A \sin \alpha t + B \cos \alpha t \). Identify the closest keyboard note to the pitch represented by this function, and the error (if any) in cents.

\[ d = 3\sqrt{2} \quad \text{(amplitude)} \]
\[ F = 165 \text{ Hz} \]
\[ \frac{\pi}{4} \quad \text{(phase shift)} \]
\[ = 12 \log_2 \left( \frac{165}{110} \right) \approx 7.02 \]
\[ \text{so 7 semitones above } A_2 \quad \text{so E}_3 \quad \text{(error: 2 cents)} \]

\[ \cos \frac{\pi}{4} = \frac{\sqrt{2}}{2} = \sin \frac{\pi}{4} \]
\[ h(t) = 3\sqrt{2} \left( \frac{\sqrt{2}}{2} \sin(330\pi t) + \frac{1}{2} \cos(330\pi t) \right) \]
\[ = 3 \sin(330\pi t) + 3 \cos(330\pi t) \]
9. On the staff system below, write the keyboard's best approximation for each prime harmonic up through 17 for the indicated note, indicating how sharp or flat (in cents) the keyboard's approximation is.

10. (a) What is the mean-tone fifth? Give its value and explain how it arises. Compare it, in cents, to the just fifth and the fifth of equal temperament.

(b) For each scale tone $\hat{1}$ to $\hat{8}$, write the fraction that expresses the ratio of that scale tone to $\hat{1}$ in the just intonation diatonic scale. Factor each fraction into primes.

Express the intervals from $\hat{2}$ to $\hat{6}$ and from $\hat{3}$ to $\hat{7}$ as a fractions and in cents. Are either of these the just fifth?