Homework 5
Math 109 / Music 109A, Spring 2003

Due Monday, April 5.

(1) Express the following interval ratios in terms of \( n \)-chromatic units, for the given \( n \). Round off to 2 digits to the right of the decimal.

(a) ratio \( \frac{5}{2} \); \( n = 19 \)

(b) ratio 3; \( n = 8 \)

(c) ratio 0.85; \( n = 13 \)

(d) ratio \( 2\pi \); \( n = 4 \) (i.e., minor thirds)

(2) Prove that there are infinitely many prime numbers. (Hint: If \( p_1, \ldots, p_n \) were a complete list of primes, consider a prime factor of \( p_1 \cdots p_n + 1 \).)

(3) Show that the functions \( f(x) = b^x \) and \( g(x) = \log_b(x) \) are group homomorphisms, and that they are inverse to each other, thereby giving isomorphisms between the groups \((\mathbb{R}, +)\) and \((\mathbb{R}^+, \cdot)\).

(4) Express the following iterations of chromatic intervals as \( r \) semitones with \( 0 \leq r < 12 \). Interpret all these iterations as operations in \( \mathbb{Z}_{12} \).

(a) the iteration of 14 and 23 semitones

(b) two fifths and a major third

(c) six fifths

(d) up three minor thirds, down six steps

(5) Given \( m \in \mathbb{Z}^+ \) and \( n \in \mathbb{Z} \), prove that \([n]\) is a generator for \( \mathbb{Z}_m \) if and only if \( \gcd(m, n) = 1 \). Interpret this as a statement about generating intervals in the modular \( m \)-chromatic scale.

(6) In the first six complete measures of Liebestraum (see accompanying excerpt) identify the sequence of progressions which goes from I to III\(^7\) then follows the circle of fifths back to I, all intermediate chords being seventh chords. Label each measure according to Roman numeral with suffix. Noting that some notes in the melody and/or arpeggio do not lie within the chord, circle all the non-chord tones.
(7) For each of these choices of \( n \), determine \( \phi(n) \) (\( \phi \) is the Euler Phi function) by listing all the generating intervals in the \( n \)-chromatic scale. Indicate which pairs of generating intervals are invers to each other and for each pair draw the circle of intervals which is based on one element of the pair in the clockwise direction, the other element of the pair in the counterclockwise direction.

(a) \( n=6 \)  \hspace{1cm} (b) \( n=5 \)  \hspace{1cm} (c) \( n=9 \)  \hspace{1cm} (d) \( n=10 \)

(8) Suppose \( G \) is a group and \( g \in G \). Show that there is a unique group homomorphism \( \varphi : \mathbb{Z} \to G \) such that \( \varphi(1) = g \).

(9) Create a twelve-tone row chart having this sequence as its original row:

\[
\begin{align*}
\text{\#} & \quad \text{\#} & \quad \text{\#} & \quad \text{\#} & \quad \text{\#} & \quad \text{\#} & \quad \text{\#} & \quad \text{\#} & \quad \text{\#} & \quad \text{\#} & \quad \text{\#} & \quad \text{\#} & \quad \text{\#} \\
\text{\#} & \quad \text{\#} & \quad \text{\#} & \quad \text{\#} & \quad \text{\#} & \quad \text{\#} & \quad \text{\#} & \quad \text{\#} & \quad \text{\#} & \quad \text{\#} & \quad \text{\#} & \quad \text{\#} & \quad \text{\#} & \quad \text{\#} & \quad \text{\#} \\
\end{align*}
\]

Write a short composition (say, \( \leq 3 \) measures) which uses only the retrograde of original row’s inversion (i.e., the left column of the chart read from bottom to top), incorporating some harmonic material. (You need not turn in a sound file for this composition, although you may do so if you wish.)

(10) Create \( n \)-tone row charts for the following choices of \( n \) and the given sequences of original rows in \( \mathbb{Z}_n \):

(a) \( n = 3 \); (\([0], [2], [1]\))

(b) \( n = 5 \); (\([0], [4], [2], [3], [1]\))

(c) \( n = 6 \); (\([0], [2], [4], [1], [3], [5]\))

(d) \( n = 7 \); (\([0], [5], [6], [3], [2], [1], [4]\))