Homework 5
Math 109 / Music 109A, Spring 2005

Due Monday, April 4.

(1) Consider the Fourier series

\[ f(t) = 1 + \sum_{k=1}^{\infty} \left[ \frac{1}{k^2} \sin(400\pi kt) + \frac{1}{k^2} \cos(400\pi kt) \right] . \]

For the \( k \)th harmonic, determine the amplitude, phase shift, and frequency.

(2) Suppose musical tone with pitch B\(_4\) has harmonics 1, 3, 5 only, with amplitudes \( \frac{1}{9} \), \( \frac{1}{25} \), respectively, and phase shifts 0, \( \pi \), \( -\frac{\pi}{2} \), respectively. Suppose also that the vertical shift \( C \) is 0. Write its Fourier series in the form \( \sum [A_k \sin(kt) + B_k \cos(kt)] \).

(3) Evaluate without a calculator by writing the argument of \( \log \) as a power of the base. Write down each step of the simplification, e.g., \( \log_3 3 \sqrt{3} = \log_3 3^{3/2} = \frac{3}{2} \log_3 3 = \frac{3}{2} \):

(a) \( \log_{10}(0.01) \)  
(b) \( \log_2 16 \)  
(c) \( \log_5 \sqrt[3]{25} \)  
(d) \( \log_c \sqrt{c^7} \)

(4) Express as a single logarithm without coefficient, i.e., in the form \( \log_b c \) (do not evaluate with a calculator):

(e) \( \log_2 5 + \log_2 3 \)  
(f) \( \log_4 7 - 2\log_4 11 \)  
(g) \( \log_3 10 + \log_9 16 \)  
(h) \( 2\log_a x^2 - \frac{1}{2} \log_a x \)

(5) Sketch the graphs of:

(a) \( f(x) = 5^x \)  
(b) \( g(x) = \log_5 x \)  
(c) \( r(x) = 3^x \)  
(d) \( s(x) = \log_3 x \)

Determine which pairs of these functions are inverse to each other, and which pairs differ by a horizontal or vertical stretch/compression. In the latter case, identify the stretch factor, justifying your answer.

(6) Using laws of exponents, prove these properties of logarithms:

\[ \log_b \frac{x}{y} = \log_b x - \log_b y \]
\[ \log_b (x^p) = p \log_b x \]
(7) Suppose \( n \in \mathbb{Z}^+ \) and we want the interval of an octave to correspond to a distance of \( n \) on a logarithmic axis parameterizing pitch. What base should we choose? Justify your answer.

(8) Convert to semitones the intervals given by the following ratios: (Round off to 2 digits to the right of the decimal.)

(a) 3 \hspace{1cm} (b) 0.8 \hspace{1cm} (c) \frac{4}{3} \hspace{1cm} (d) \sqrt[3]{2} \hspace{1cm} (e) \ e

(9) Convert to cents the intervals given by the following ratios, rounding off to the nearest whole cent:

(f) 1.25 \hspace{1cm} (g) 1.1 \hspace{1cm} (h) \frac{7}{4} \hspace{1cm} (i) \frac{2}{3} \hspace{1cm} (j) \pi

(10) Write on the staff the note which best approximates the frequency having the given interval ratio \( r \) from the given note:

(a) \hspace{1cm} (b) \hspace{1cm} (c) \hspace{1cm} (d)

\[
\begin{align*}
&\text{\includegraphics{note_a}} \hspace{2cm} \text{\includegraphics{note_b}} \hspace{2cm} \text{\includegraphics{note_c}} \hspace{2cm} \text{\includegraphics{note_d}} \\
&r = 3 \hspace{2cm} r = \frac{5}{3} \hspace{2cm} r = 2.3 \hspace{2cm} r = \pi^{-1}
\end{align*}
\]