1. (a) \(1 + i = \sqrt{2}(\cos(\pi/4) + i\sin(\pi/4)) = \sqrt{2}e^{\pi i/4}\), hence

\[(1 + i)^8 = 2^8/2^{8\pi i/4} = 16e^{2\pi i} = 16.\]

(b) \[\frac{1 + i\sqrt{3}}{1 - i\sqrt{3}} = \frac{1 + i\sqrt{3}}{1 - (i\sqrt{3})^2} = 1/4 + i\sqrt{3}/4 = \frac{1}{2}(\cos(\pi i/3) + \sin(\pi i/3)) = \frac{1}{2}e^{\pi i/3}.\]

(c) Since \(i = e^{\pi i/2}\), \(\sqrt{i}\) has values \(e^{\pi i/4 + 2\pi ik/2}\), \(k = 0, 1\), that is, \(e^{\pi i/4} = 1/\sqrt{2} + i/\sqrt{2}\) and \(e^{5\pi i/4} = -1/\sqrt{2} - i/\sqrt{2}\).

(d) Since \(-1 = e^{\pi i}\), \(\sqrt[4]{-1}\) has values \(e^{\pi i/4 + 2\pi ik/4}\), \(k = 0, 1, 2, 3\). These are \(\{e^{\pi i/4}, e^{3\pi i/4}, e^{5\pi i/4}, e^{7\pi i/4}\}\), or

\(\{1/\sqrt{2} + i/\sqrt{2}, -1/\sqrt{2} + i/\sqrt{2}, -1/\sqrt{2} - i/\sqrt{2}, 1/\sqrt{2} - i/\sqrt{2}\}\).
2. Since

\[ \cos 4\theta + i \sin 4\theta = (\cos \theta + i \sin \theta)^4 \]

\[ = \cos^4 \theta + 4i \cos^3 \theta \sin \theta - 6 \cos^2 \theta \sin^2 \theta + 4i \cos \theta \sin^3 \theta + \sin^4 \theta, \]

it follows that

\[ \cos 4\theta = \cos^4 \theta - 6 \cos^2 \theta \sin^2 \theta + \sin^4 \theta, \]

\[ \sin 4\theta = 4 \cos^3 \theta \sin \theta - 4 \cos \theta \sin^3 \theta. \]

3. The identity follows from

\[ n \sum_{i=1}^{n} z_i w_i \]

\[ = n \sum_{i=1}^{n} \bar{z}_i \bar{w}_i = \sum_{i,j=1}^{n} z_i \bar{z}_j w_i \bar{w}_j \]

and

\[ \sum_{i=1}^{n} |z_i|^2 \sum_{i=1}^{n} |w_i|^2 - \sum_{1 \leq i < j \leq n} |z_i \bar{w}_j - z_j \bar{w}_i|^2 \]

\[ = \sum_{i=1}^{n} z_i \bar{z}_i \sum_{j=1}^{n} w_j \bar{w}_j - \sum_{1 \leq i < j \leq n} (z_i \bar{w}_j - z_j \bar{w}_i)(\bar{z}_i w_j - \bar{z}_j w_i) \]

\[ = \sum_{i,j=1}^{n} z_i \bar{z}_i w_j \bar{w}_j - \sum_{1 \leq i < j \leq n} (z_i \bar{z}_i w_j \bar{w}_j + z_j \bar{z}_j w_i \bar{w}_i) + \sum_{1 \leq i < j \leq n} (z_i \bar{z}_j w_i \bar{w}_j + z_j \bar{z}_i w_i \bar{w}_i) \]

\[ = \sum_{i,j=1}^{n} z_i \bar{z}_i w_j \bar{w}_j - \sum_{i \neq j} z_i \bar{z}_i w_j \bar{w}_j + \sum_{i \neq j} z_i \bar{z}_j w_i \bar{w}_j = \sum_{i=1}^{n} z_i \bar{z}_i w_i \bar{z}_i + \sum_{i \neq j} z_i \bar{z}_j w_i \bar{w}_j \]

\[ = \sum_{i,j=1}^{n} z_i \bar{z}_j w_i \bar{z}_j. \]

4. Since equilateral triangles are preserved under linear transformations \( z \mapsto az + b \), the same should be true for the identity

\[ z_1^2 + z_2^2 + z_3^2 = z_1 z_2 + z_1 z_3 + z_2 z_3. \]
Let us check this: when $z_i$ is replaced with $az_i$, both sides of (1) are multiplied by $a^2$. Furthermore, under the transformation $z_i \mapsto z_i + b$ same additional terms appear on both sides of (1), namely, $2b(z_1 + z_2 + z_3) + 3b^2$. Hence (1) is linearly invariant.

Suppose that at least two of the numbers $z_1, z_2, z_3$ are distinct. By an appropriate linear transformation we can make $z_1 = 0$ and $z_2 = 1$. Then equation (1) takes form $1 + z_3^2 = z_3$, and its roots are $(1 \pm i\sqrt{3})/2$. Since
\[ |(1 \pm i\sqrt{3})/2| = 1 = |(1 \pm i\sqrt{3})/2 - 1|, \]
these are precisely the vertices of two equilateral triangles that have $[0, 1]$ as their side.

Finally, if $z_1 = z_2 = z_3$, then (1) holds, and we can consider these points as the vertices of a degenerate equilateral triangle.

5. Suppose that $P = (X, Y, Z)$ corresponds to $z \in \mathbb{C} \setminus \{0\}$ under stereographic projection. Then $z = (X + iY)/(1 - Z)$, and the antipodal point $(-X, -Y, -Z)$ corresponds to
\[
\frac{-X - iY}{1 + Z} = \frac{-(X + iY)(X - iY)}{(1 + Z)(X - iY)} = -\frac{X^2 + Y^2}{(1 + Z)(X - iY)} = -\frac{1 - Z^2}{(1 + Z)(X - iY)} = \frac{1 - Z}{X - iY} = -1/\bar{z}.
\]

It remains to observe that $\infty$ and 0 correspond to antipodal points $(0, 0, \pm 1)$. 