Homework 3, Math 421, Fall 2003

Due Friday, September 26.

1. Show that if \( h(z) \) is a complex valued harmonic function, i.e., it satisfies Laplace’s equation, such that \( z h(z) \) is also harmonic, then \( h(z) \) is holomorphic.

2. Show that \( \log |z| \) is harmonic on the punctured plane \( \mathbb{C} \setminus \{0\} \) but has no conjugate harmonic function on \( \mathbb{C} \setminus \{0\} \), though it does have a conjugate harmonic function on the slit plane \( \mathbb{C} \setminus (-\infty, 0]\).

3. Let \( A \) and \( B \) be positive real numbers with \( B < \pi \). Find a conformal mapping from the strip \(-A < \text{Re} \, z < A\) onto the wedge \(-B < \text{Arg} \, z < B\).

4. Show that the image of a straight line under the inversion \( f(z) = \frac{1}{z} \) is a straight line or a circle, depending on whether the line passes through the origin.

5. In any group \( G \), two elements \( g \) and \( h \) are called conjugate if there exists \( a \in G \) such that \( h = aga^{-1} \). This is easily seen to be an equivalence relation on \( G \). (You don’t need to prove this.) This equivalence classes are called conjugacy classes.

Classify conjugacy classes in the group of fractional linear transformations \( f \) by establishing the following:

(a) If \( f \) is not the identity has either one or two fixed points.

(b) If \( f \) has two fixed points, then \( f \) is conjugate to a dilation \( z \mapsto az \) with \( a \neq 0, 1 \). Is \( a \) unique?

(c) If \( f \) has one fixed point then \( f \) is conjugate to the translation \( z \mapsto z + 1 \).