1. Suppose that \( p(z) = \sum_{k=0}^{n} a_k z^k \neq 0 \) for all \( z \in \mathbb{C} \) (here \( n > 0 \) and \( a_n \neq 0 \)). Then \( 1/p(z) \) is holomorphic in \( z \). When \( |z| \) is sufficiently large, we have
\[
|p(z)| = \left| \sum_{k=0}^{n} a_k z^k \right| \geq |a_n z^n| - \sum_{k=0}^{n-1} |a_k| |z|^n \geq |a_n||z|^n - |z|^{n-1} \sum_{k=0}^{n-1} |a_k| > \frac{|a_n|}{2} |z|^n.
\]
Applying the Maximum Principle to \( 1/p \) in the disk \( \{ z : |z| < R \} \) with large \( R \), obtain \( |1/p(z)| \leq 2/(|a_n|R^n), \ |z| \leq R \). Since \( R \) can be arbitrarily large, \( 1/p(z) \) must vanish identically, which is absurd.

2. Let \( C = \sup_{z \in H} |f(z)| \), where \( H = \{ z : \text{Re} \ z > 0 \} \). For \( R > 1 \) let \( \Omega_R = H \cap \{ z : |z+1| < R \} \). Choose \( \varepsilon > 0 \) arbitrarily. For all sufficiently large \( R \) the inequality \( R^{-\varepsilon}C \leq M \) holds. Now
\[
|(z+1)^{-\varepsilon}f(z)| \leq \begin{cases} R^{-\varepsilon}C \leq M, & z \in \partial \Omega_R \cap H; \\ M, & z \in \partial \Omega_R \setminus H. \end{cases}
\]
By the Maximum Principle \( |(z+1)^{-\varepsilon}f(z)| \leq M \) in \( \Omega_R \). Since \( R \) can be arbitrarily large, \( |(z+1)^{-\varepsilon}f(z)| \leq M \) in \( H \). Letting \( \varepsilon \to 0 \), obtain \( |f(z)| \leq M \) in \( H \).

3. (a) Suppose that \( F \) is a primitive of \( f \). Let \( \gamma : [0,1] \to D \) be a closed [rectifiable] path. By the Fundamental Theorem of Calculus
\[
\int_{\gamma} f(z)dz = \int_{0}^{1} f(\gamma(t))d\gamma(t) = F(\gamma(1)) - F(\gamma(0)) = 0
\]
\( (\gamma(1) = \gamma(0) \) because \( \gamma \) is closed).

(b) Conversely, suppose that \( \int_{\gamma} f(z)dz = 0 \) for any closed, piecewise linear path \( \gamma \) in \( D \). Fix \( z_0 \in D \). By our assumption the integral \( \int_{\gamma} f(\zeta)d\zeta \) does not depend on \( \gamma \) as long as \( \gamma \) is a piecewise linear path from \( z_0 \) to a certain point \( z \in D \). Let us denote the value of this integral by \( F(z) \). It remains to prove that \( F' = f \).

Fix \( z \in D \) and \( \varepsilon > 0 \). There exists \( \delta > 0 \) such that \( B = \{ z' : |z' - z| < \delta \} \subset D \) and \( |f(z') - f(z)| < \varepsilon \) for all \( z' \in B \). For any \( z' \in B \) the difference \( F(z') - F(z) \) is equal to \( \int_{z}^{z'} f(\zeta)d\zeta \), where \( I \) is the segment with endpoints \( z \) and \( z' \). Therefore,
\[
F(z') - F(z) = \int_{0}^{1} f(z + t(z' - z))(z' - z)dt = f(z)(z' - z) + \int_{0}^{1} (f(z + t(z' - z)) - f(z))(z' - z)dt.
\]
When \( t \in [0,1], z + t(z' - z) \in B \). It follows that
\[
\left| \int_{0}^{1} (f(z + t(z' - z)) - f(z))(z' - z)dt \right| \leq \int_{0}^{1} |f(z + t(z' - z)) - f(z)||z' - z|dt < \varepsilon |z' - z|,
\]
which implies
\[
\frac{|F(z') - F(z)|}{z' - z} - f(z) < \varepsilon, \quad \forall |z' - z| < \delta.
\]
Thus \( F'(z) \) exists and is equal to \( f(z) \).
4. (a) Let $C_+ = \{e^{i\theta} : 0 < \theta < \pi\}$ and $C_- = \{e^{i\theta} : \pi < \theta < 2\pi\}$ be half-circles (oriented counterclockwise). Applying Cauchy’s theorem to $f(z)^2$ in two half-disks, obtain

$$\int_{-1}^{1} f(x)^2 dx = -\int_{C_+} f(z)^2 dz = \int_{C_-} f(z)^2 dz.$$ 

Hence

$$\int_{-1}^{1} f(x)^2 dx = \frac{1}{2} \left\{ \left| \int_{C_+} f(z)^2 dz \right| + \left| \int_{C_-} f(z)^2 dz \right| \right\} \leq \frac{1}{2} \left\{ \int_0^\pi |f(e^{i\theta})|^2 d\theta + \int_0^\pi |f(e^{i\theta})|^2 d\theta \right\} = \frac{1}{2} \int_0^{2\pi} |f(e^{i\theta})|^2 d\theta.$$ 

On the other hand,

$$\int_0^{2\pi} |f(e^{i\theta})|^2 d\theta = \int_0^{2\pi} \sum_{k=0}^n c_k e^{i(k-1)\theta} d\theta = \int_0^{2\pi} \sum_{k=0}^n c_k c_l e^{i(k-l)\theta} d\theta = 2\pi \sum_{k=0}^n |c_k|^2. \quad \text{(1)}$$

(The identity (1) is true for any $c_k \in \mathbb{C}.$) \hfill \square

(b) Now $c_k = a_k + ib_k$ with $a_k, b_k \in \mathbb{R}$. Applying part (a) to $p(z) = \sum_{k=0}^n a_k z^k$ and $q(z) = \sum_{k=0}^n b_k z^k$, and then applying (1) to $f$, we obtain

$$\int_{-1}^{1} |f(x)|^2 dx = \int_{-1}^{1} (p(x)^2 + q(x)^2) dx \leq \pi \sum_{k=0}^n (a_k^2 + b_k^2) = \pi \sum_{k=0}^n |c_k|^2 = \frac{1}{2} \int_0^{2\pi} |f(e^{i\theta})|^2 d\theta. \quad \square$$

5. (a) There are 3 singular point inside of the circle $|z| = 2$, but just one outside of it (namely, $\infty$). The residue at $\infty$ is equal to the Laurent coefficient of degree $-1$ with the opposite sign. When $|z|$ is large, $(z^3 - 1)^{-1} = z^{-3} (1 - z^{-3})^{-1} = z^{-3} \sum_{k=0}^\infty z^{-3k} = \sum_{k=0}^\infty z^{-3k-3}$, so the residue at $\infty$ is zero. Thus $\oint_{|z|=2} (z^3 - 1)^{-1} dz = -2\pi i \text{res}_{z=\infty} (z^3 - 1)^{-1} = 0.$

Another approach: let $A_R = \{z : 2 < |z| < R\}$, $R > 2$. Since $1/(z^3 - 1)$ is holomorphic in $A_R$, we have $\int_{\partial A_R} (z^3 - 1)^{-1} dz = 0$. Hence $\oint_{|z|=2} (z^3 - 1)^{-1} dz = \oint_{|z|=R} (z^3 - 1)^{-1} dz$. But

$$\left| \oint_{|z|=R} (z^3 - 1)^{-1} dz \right| \leq \oint_{|z|=R} |z^3 - 1|^{-1} |dz| \leq 2\pi R/(R^3 - 1) \to 0, \quad R \to \infty.$$ 

This implies $\oint_{|z|=2} (z^3 - 1)^{-1} dz = 0.$

(b) Since $z^{-1} \sin z$ is holomorphic in $\mathbb{C}$, the integral is equal to 0 by Cauchy’s theorem.

(c) No complex analysis here:

$$\oint_{|z|=1} |z - 1||dz| = \int_0^{2\pi} |e^{i\theta} - 1| d\theta = \int_0^{2\pi} \sqrt{(\cos \theta - 1)^2 + \sin^2 \theta} d\theta$$

$$= \int_0^{2\pi} \sqrt{2 - 2\cos \theta} d\theta = 2 \int_0^{2\pi} \sin(\theta/2) d\theta = -4(\cos \pi - \cos 0) = 8.$$