1. Let $U \subset \mathbb{C}$ be a [non-empty] open set such that $f(\mathbb{C}) \cap U = \emptyset$. Choose $w_0 \in U$; there is $r > 0$ such that $\{w : |w - w_0| < r\} \subset U$. Consequently, $|f(z) - w_0| \geq r$ for all $z \in \mathbb{C}$. Now the function $z \mapsto 1/(f(z) - w_0)$ is bounded by $1/r$ and is holomorphic in $\mathbb{C}$. By Liouville’s theorem $1/(f(z) - w_0) \equiv A$ for some $A \in \mathbb{C}$. Hence $f(z) \equiv w_0 + 1/A$. \hfill □

2. Let $R = \left(\limsup_{k \to \infty} |a_k|^{1/k}\right)^{-1}$; our goal is to prove that the series $\sum a_k z^k$ converges when $|z| < R$ and diverges when $|z| > R$.

   (a) Suppose that $|z| < R$. In this case $\limsup_{k \to \infty} |a_k|^{1/k} = 1/R < \infty$. Choose $\varepsilon > 0$ so that $|z|(1/R + \varepsilon) < 1$. Now for sufficiently large $k$
   
   $$|a_k z^k| = (|a_k|^{1/k}|z|)^k \leq ((1/R + \varepsilon)|z|)^k.$$

   Being majorized by a convergent geometric series, $\sum a_k z^k$ converges.

   (b) Suppose that $|z| > R$. In this case $\limsup_{k \to \infty} |a_k|^{1/k} = 1/R > 0$. Choose $\varepsilon > 0$ so that $|z|(1/R - \varepsilon) > 1$. Now for infinitely many values of $k$
   
   $$|a_k z^k| = (|a_k|^{1/k}|z|)^k \geq ((1/R - \varepsilon)|z|)^k \to \infty.$$

   Since the terms of $\sum a_k z^k$ are not bounded, the series diverges. \hfill □

3. (a) $R = \left(\limsup_{k \to \infty} |3^{1/k}|\right)^{-1} = 1/3$.

   (b) $R = \left(\limsup_{k \to \infty} |2^k/(k^2 + k)|^{1/(2k)}\right)^{-1} = 2^{-1/2} \liminf_{k \to \infty} (k^2 + k)^{1/(2k)}$

   $$= 2^{-1/2} \liminf_{k \to \infty} \exp\left\{\log(k^2 + k)/(2k)\right\} = 1/\sqrt{2}.$$

   (c) $R = \left(\limsup_{k \to \infty} |k!/(k/k)|^{1/k}\right)^{-1} = (\limsup_{k \to \infty} (k!)^{1/k}/k)^{-1} = (e^{-1})^{-1} = e$.

   because

   $$\lim_{k \to \infty} \log((k!)^{1/k}/k) = \lim_{k \to \infty} \frac{1}{k} \sum_{j=1}^{k} \log j - \log k = \lim_{k \to \infty} \frac{1}{k} \sum_{j=1}^{k} \log(j/k)$$

   $$= \int_0^1 \log x \, dx = x \log x - x \bigg|_0^1 = -1.$$

   □