1. (a) Since the function \( h(z) = f(z)g(z) \) is holomorphic in the disk \( \{ z : |z| < \min(R, S) \} \), it is represented by a power series \( \sum_{k=0}^{\infty} d_k z^k \) which converges in this disk. Here

\[
\frac{d_k}{k!} = \frac{h^{(k)}(0)}{k!} = \frac{1}{k!} \sum_{j=0}^{k} \binom{k}{j} f^{(j)}(0) g^{(k-j)}(0) = \sum_{j=0}^{k} \frac{1}{j!(k-j)!} f^{(j)}(0) g^{(k-j)}(0) = \sum_{j=0}^{k} a_j b_{k-j},
\]
as required.

Another approach, using some background on infinite series. Given two convergent series \( \sum_{k=0}^{\infty} \alpha_k = A \) and \( \sum_{k=0}^{\infty} \beta_k = B \), one defines their product as \( \sum_{k=0}^{\infty} \gamma_k \), where \( \gamma_k = \sum_{j=0}^{k} \alpha_j \beta_{k-j} \). The series \( \sum_{k=0}^{n} \gamma_k \) may diverge. However, if either \( \sum_{k=0}^{\infty} \alpha_k \) or \( \sum_{k=0}^{\infty} \beta_k \) converges absolutely, then \( \sum_{k=0}^{\infty} \gamma_k = AB \) (see Rudin’s *Principles of Mathematical Analysis*).

Now if \( |z| < \min(R, S) \), then both \( \sum_{k=0}^{\infty} a_k z^k \) and \( \sum_{k=0}^{\infty} b_k z^k \) converge absolutely, hence \( \sum_{k=0}^{\infty} c_k z^k = f(z)g(z) \). □

(b) \( f(z) = (z - 1)/(z - 2) \), \( g(z) = (z - 2)/(z - 1) \).

2. We have

\[
\frac{z}{2} \cot(z/2) = \frac{z}{2} \left( e^{iz/2} + e^{-iz/2}/2 \right)/2 = \frac{iz e^{iz} + 1}{2 e^{iz} - 1} = \frac{iz}{2} + \frac{iz}{e^{iz} - 1}.
\]

The function \( e^{iz} - 1 \) has zeroes \( 2\pi k, k \in \mathbb{Z} \). Hence the function \( \frac{z}{2} \cot(z/2) \) has simple poles \( 2\pi k, k \in \mathbb{Z} \setminus \{0\} \). Therefore, its Taylor series at 0 has radius of convergence \( 2\pi \).

Now

\[
\frac{iz}{e^{iz} - 1} = 1 - \frac{iz}{2} \sum_{n=1}^{\infty} B_n \frac{z^{2n}}{(2n)!},
\]

while

\[
\frac{e^{iz} - 1}{iz} = \sum_{n=0}^{\infty} \frac{(iz)^n}{(n+1)!}.
\]

By virtue of #1, formal multiplication of the two series produces \( 1 + 0 + 0 + \ldots \). Let us look at the coefficient of \( z^{2n} \) \( (n > 1) \) in the product:

\[
0 = \frac{z^{2n}}{(2n+1)!} - \frac{i z^{2n-1}}{2 \cdot (2n)!} - \sum_{k=1}^{n} \frac{B_k}{(2k)!} \frac{i^{2n-2k}}{(2n-2k+1)!}
= \frac{(-1)^n}{(2n+1)!} - \frac{(-1)^n}{2(2n)!} \sum_{k=1}^{n} \frac{B_k}{(2k)!} \frac{(-1)^{n-k}}{(2n-2k+1)!}.
\]
It follows that

$$(-1)^n B_n = (2n)! \left\{ \frac{1}{(2n+1)!} - \frac{1}{2(2n)!} - \sum_{k=1}^{n-1} \frac{(-1)^k B_k}{(2k)!(2n-2k+1)!} \right\}$$

$$= \frac{1}{2n+1} \left\{ \frac{1}{2} - n - \sum_{k=1}^{n-1} (-1)^k B_k \left( \frac{2n+1}{2k} \right) \right\}.$$

Using this recursive formula, one can compute the first five Bernoulli numbers without a calculator:

$$-B_1 = 1 \left( \frac{1}{3} \frac{1}{2} - 1 \right) = -\frac{1}{6};$$

$$B_2 = \frac{1}{5} \left( \frac{1}{2} - 2 - \frac{1}{6} \right) \left( \frac{5}{2} \right) = \frac{1}{30};$$

$$-B_3 = \frac{1}{7} \left( \frac{1}{2} - 3 - \frac{1}{6} \right) \left( \frac{7}{2} \right) - \frac{1}{30} \left( \frac{7}{4} \right) = -\frac{1}{42};$$

$$B_4 = \frac{1}{9} \left( \frac{1}{2} - 4 - \frac{1}{6} \right) \left( \frac{9}{2} \right) - \frac{1}{30} \left( \frac{9}{4} \right) - \left( \frac{1}{42} \right) \left( \frac{9}{6} \right) = \frac{1}{30};$$

$$-B_5 = \frac{1}{11} \left( \frac{1}{2} - 5 - \frac{1}{6} \right) \left( \frac{11}{2} \right) - \frac{1}{30} \left( \frac{11}{4} \right) - \left( \frac{1}{42} \right) \left( \frac{11}{6} \right) - \frac{1}{30} \left( \frac{11}{8} \right) \right\} = -\frac{5}{66}.$$  

So, $B_1 = 1/6$, $B_2 = 1/30$, $B_3 = 1/42$, $B_4 = 1/30$, $B_5 = 5/66$.  

3. Let $0 < r < \rho$. Since $f_m \to f$ uniformly on $\{ z : |z| \leq r \}$, it follows that

$$|a_{k,m} - a_k| = \frac{1}{2\pi} \left| \int_{|z|=r} \frac{f_m(z)-f(z)}{z^{k+1}} \frac{dz}{z^{k+1}} \right| \leq \frac{1}{2\pi r^{k+1}} \int_{|z|=r} |f_m(z)-f(z)||dz| \xrightarrow{m \to \infty} 0.$$ 

4. When $|z-z_0|$ is sufficiently small, we have

$$f(z) = f(z_0) = \sum_{n=0}^{\infty} c_n (z-z_0)^n = (z-z_0)^N \left( c_N + \sum_{n=N+1}^{\infty} c_n (z-z_0)^{n-N} \right),$$

where $c_N \neq 0$. By continuity $c_N + \sum_{n=N+1}^{\infty} c_n (z-z_0)^{n-N} \neq 0$ in some open disk $U$ containing $z_0$. Let $h(z)$ be a holomorphic branch of $(c_N + \sum_{n=N+1}^{\infty} c_n (z-z_0)^{n-N})^{1/N}$ in $U$. Since $h(z_0) \neq 0$, the function $g(z) = (z-z_0)h(z)$ satisfies $g'(z_0) = h(z_0) \neq 0$. By construction $f(z) = g(z)^N$ in $U$.  

5. Let $U \subset D$ be an open set. To prove that $f(U)$ is open, we have to show that for any $z_0 \in U$ there exists $\varepsilon > 0$ such that $B_\varepsilon (f(z_0)) := \{ w : |w-f(z_0)| < \varepsilon \} \subset f(U)$. First suppose that $f'(z_0) \neq 0$. Then $f$ has a continuous inverse $g$ in some $B_\delta (f(z_0))$. Since $U$ is open, there is $\sigma < \delta$ such that $g(B_\sigma (f(z_0))) \subset U$. Hence $B_\sigma (f(z_0)) \subset f(U)$.

Now suppose $f'(z_0) = 0$. Since $f$ is nonconstant, by #4 there exists an integer $N > 0$ such that $f(z) - f(z_0) = g(z)^N$ in $B_r(z_0)$, where $r > 0$ and $g'(z_0) \neq 0$. By above there is $\sigma > 0$ such that $B_\sigma (0) \subset g(U)$ (note that $g(z_0) = 0$). Let $\varepsilon = \sigma^N$. Since $B_\varepsilon (0) \subset g^N(U)$, it follows that $B_\varepsilon (f(z_0)) \subset f(U)$.  

□