Homework 8, Math 421, Fall 2003

Due Wednesday, November 19.

1. Suppose $f(z)$ is meromorphic on $\mathbb{C} \cup \{\infty\}$. For $\omega \in \mathbb{C} \cup \{\infty\}$ define

$$\text{ord}_\omega(f) = \begin{cases} 
N & \text{if } f \text{ has a zero of order } N \text{ at } \omega; \\
-N & \text{if } f \text{ has a pole of order } N \text{ at } \omega; \\
0 & \text{otherwise.}
\end{cases}$$

Show that

$$\sum_{\omega \in \mathbb{C} \cup \{\infty\}} \text{ord}_\omega(f) = 0.$$ 

2. A meromorphic function $f(z)$ of $\mathbb{C}$ is called doubly periodic if the group of periods for $f$ is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$. Show that there are no entire doubly periodic functions.

3. Let $V$ be the vector space (over $\mathbb{C}$) of functions that are holomorphic on $\mathbb{C} \cup \{\infty\}$ except possibly at 0 and $i$, where they have poles of order of order $\leq 2$. Determine the dimension of $V$ by finding an explicit vector space basis for $V$.

4. Expand $\tan 2\pi z$ as a series $\sum_{k=-\infty}^{\infty} a_k e^{2\pi ikz}$ converging on the upper half plane.

5. Suppose 1 is a period of a double periodic function, and that there are no periods $w$ with $0 < |w| < 1$. Describe the possible configurations of periods $w$ with $|w| = 1$. 