In the previous lecture we introduced the method of partial fractions and stated that the <u>partial fraction decomposition</u> of a rational function $\frac{P(x)}{Q(x)}$ (*where degree $P(x)$ is smaller than degree $Q(x)$*) depends on the "mix" of factors when $Q(x)$ is factored completely.

The situation when $Q(x)$ has only linear factors, perhaps with some repeated was discussed in the preceding lecture. (Cases I and II in the textbook): in summary

> if a linear factor $(ax+b)$ is repeated n times
> then put n fractions $\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + ... + \frac{A_n}{(ax+b)^n}$ into the partial fraction decomposition.

Q1 What is the form for the partial fraction decomposition of $\frac{1}{x^3(x+1)(x+2)^2}$

A) $\frac{A}{x} + \frac{B}{x+1} + \frac{C}{x+2}$

B) $\frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{D}{x+1} + \frac{E}{x+2}$

C) $\frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{D}{x+1} + \frac{E}{x+2} + \frac{Fx+G}{(x+2)^2}$

D) $\boldsymbol{\frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{D}{x+1} + \frac{E}{x+2} + \frac{F}{(x+2)^2}}$

E) $\frac{A}{x} + \frac{Bx+C}{x^2} + \frac{Dx^2+Ex+F}{x^3} + \frac{D}{x+1} + \frac{E}{x+2} + \frac{Fx+G}{(x+2)^2}$

<u>Answer</u>

> The linear factor x is repeated 3 times : $x^3 = x \cdot x \cdot x$ so it contributes $\frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3}$ to the partial fraction decomposition.
>
> The linear factor $(x+1)$ is "repeated" only one time, so it contributes $\frac{D}{x+1}$ to the parti9al fraction decomposition.
>
> The linear factor $(x+2)$ is repeated 2 times, so it contributes $\frac{E}{x+2} + \frac{F}{(x+2)^2}$ to the partial fraction decomposition
>
> Combining all these gives the partial fraction decomposition: answer D

The following summary includes the preceding comments and also describes what happens when irreduciable quadratic factors occur in $Q(x)$.

For a rational function $\frac{P(x)}{Q(x)}$ where degree of $P(x)$ $<$ degree of $Q(x)$

Completely factor $Q(x) = (\ )\ (\ )\ (\ ) \ldots\ (\ )$

$\uparrow$

each factor is either	linear	$(ax+b)$ <u>or</u>
	irreducible quadratic	(ax^2+bx+c)

To write the <u>partial fraction decomposition</u> (pfd) for $\frac{P(x)}{Q(x)}$

$$= \frac{A}{(\ )} + \frac{B}{(\ )} + \frac{C}{(\ )} + \ldots \text{ (however many fractions it takes)}$$

$$\begin{cases} \text{for each } \underline{\text{nonrepeated}}\ (ax+b), \text{ put a fraction } \frac{A}{ax+b} \text{ into the pf}d \\ \text{for each } \underline{\text{nonrepeated}} \text{ (irreducible) } (ax^2+bx+c), \text{ put a fraction } \frac{Ax+B}{ax^2+bx+c} \text{ into the pfd} \end{cases}$$

<u>Example</u> For $\frac{1}{(x+2)^2(x^2+x+1)(x^2+x+2)^2}$, the for for the pfd is

$$\frac{A}{x+2} + \frac{B}{(x+2)^2} + \frac{Cx+D}{x^2+x+1} + \frac{Ex+F}{x^2+x+2} + \frac{Gx+H}{(x^2+x+2)^2}$$

We will illustrate a case where there is an irreduicable quadratic factor. The TAs will do an example with a repeated irreducible factor in the discussion section tomorrow.

But first: when irreducible quadratic denominators occur, the following integral often comes up. It's useful to have a genera formula for it>

Q1 Find $\int \frac{1}{x^2+a^2}\,dx$

(*all answers below should include "$+C$" -- omitted here to save space*)

A) $\arctan(x^2+a^2)$ B) $a\arctan(\frac{x}{a})$ C) $\frac{1}{a^2}\arctan x$

D) $\frac{1}{a}\arctan(\frac{x}{a})$ E) $x^2\arctan(\frac{x}{a})$

<u>Answer</u> Let $x = a\tan\theta$, $dx = a\sec^2\theta\,d\theta$. Then

$$\int \frac{1}{x^2+a^2}\,dx = \int \frac{1}{a^2\tan^2\theta + a^2} a\sec^2\theta\,d\theta = \frac{1}{a}\int \frac{\sec^2\theta}{\sec^2\theta}d\theta = \frac{1}{a}\int 1\,d\theta = \frac{\theta}{a} + C = \frac{1}{a}\arctan(\frac{x}{a}) + C$$

Example $\int \frac{x+2}{(x+1)(x^2+x+1)}dx$

$\frac{x+2}{(x+1)(x^2+x+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+x+1}$. Multiply both sides by $(x+1)(x^2+x-1)$

$$\begin{aligned} x+2 &= A(x^2+x+1) + (Bx+C)(x+1) \\ &= (A+B)x^2 + (A+B+C)x + (A+C) \end{aligned}$$

so we need $\begin{cases} A+\ B & = 0 \\ A+B+C & = 1 \\ A+\quad +C & = 2 \end{cases}$ and solving gives $A = C = 1,\ B = -1$

so

$$\int \frac{x+2}{(x+1)(x^2+x+1)}dx = \int \frac{1}{x+1} + \frac{1-x}{x^2+x+1}dx = \int \frac{1}{x+1}dx + \int \frac{1-x}{x^2+x+1}dx$$

$$= \ln|x\ +1| + \int \frac{1-x}{x^2+x+1}dx$$

For $\int \frac{1-x}{x^2+x+1}dx$:

compelete the square $\int \frac{1-x}{x^2+x+1}dx = \int \frac{1-x}{(x+\frac{1}{2})^2+\frac{3}{4}}dx$

substitute $u = x + \frac{1}{2},\ du = dx,\ 1 - x = \frac{3}{2} - u$

$$= \int \frac{\frac{3}{2}-u}{u^2+\frac{3}{4}}\,du$$

split into two integrals

$$= \int \frac{\frac{3}{2}}{u^2+\frac{3}{4}}\,du - \int \frac{u}{u^2+\frac{3}{4}}\,du$$

<u>For the first integral</u>, use $\int \frac{1}{u^2+a^2}\,du = \frac{1}{a}\arctan(\frac{u}{a})$ (from Q2, above)

$$\int \frac{\frac{3}{2}}{u^2+\frac{3}{4}}\,du = \frac{3}{2}\int \frac{1}{u^2+(\frac{\sqrt{3}}{2})^2}du = \frac{3}{2}\cdot\frac{1}{(\frac{\sqrt{3}}{2})}\arctan(\frac{\mathrm{u}}{\frac{\sqrt{3}}{2}})$$

$$= \sqrt{3}\arctan(\frac{2u}{\sqrt{3}}) = \sqrt{3}\arctan(\frac{2(x+\frac{1}{2})}{\sqrt{3}}) = \sqrt{3}\arctan(\frac{2x+1)}{\sqrt{3}})$$

<u>For the second integral</u>, let $t = u^2 + \frac{3}{4},\ \frac{1}{2}dt = u\,du$, so that

$$\int \frac{u}{u^2+\frac{3}{4}}\,du = \frac{1}{2}\int \frac{1}{t}dt = \frac{1}{2}\ \ln|t| = \frac{1}{2}\ln\left(u^2+\frac{3}{4}\right)$$

$$= \frac{1}{2}\ln\left(\ (x+\frac{1}{2})^2+\frac{3}{4}\right) = \frac{1}{2}\ln(x^2+x+1)$$

<u>Putting all the pieces together:</u>

$$\int \frac{x+2}{(x+1)(x^2+x+1)}dx = \int \frac{1}{x+1} + \frac{1-x}{x^2+x+1}dx = \int \frac{1}{x+1}dx + \int \frac{1-x}{x^2+x+1}dx$$

$$= \ln|x\ +1| + \int \frac{1-x}{x^2+x+1}dx$$

$$= \ln|x\ +1| + \sqrt{3}\arctan\left(\frac{2x+1)}{\sqrt{3}}\right) - \frac{1}{2}\ln(x^2+x+1) + C$$