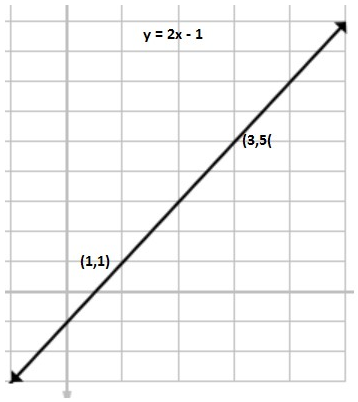
(Text, Section 8.1)

Q1: Find the length of the graph of $y = f(x) = 2x - 1$ for $1 \le x \le 3$.
(*No calculus needed*)

A) 2 B) $\sqrt{3}$ C) 4 D) $4\sqrt{3}$ **E) $2\sqrt{5}$**



<u>Answer</u> We want the length L of the straight line segment joining $(1, 1)$ to $(3, 5)$.
By the distance formula from precalculus, $L = \sqrt{(3-1)^2 + (5-1)^2} = \sqrt{20} = 2\sqrt{5}$

We briefly reviewed the Mean Value Theorem (used later in the lecture to find an integral that givesus the length of part of the graph of a function. The following formulas were derived in the lecture (and are explained in the text): that's not repeated here.

In the lecture and in the textbook, there was an explanation of how to get an integral that represents the <u>length of a curve</u> $y = f(x)$, $a \le x \le b$ (or $x = g(y)$, $c \le y \le d$)

The "piece" of the graph is sometimes referred to, loosely, as an "arc along the graph" so that the length is sometimes called <u>arc length.</u>

arc length $L = \int_a^b \sqrt{1 + (f'(x))^2}\, dx$	OR	$L = \int_c^d \sqrt{1 + (g'(y))^2}\, dy$
$= \int_a^b \sqrt{1 + (\frac{dy}{dx})^2}\, dx$	OR	$= \int_c^d \sqrt{1 + (\frac{dx}{dy})^2}\, dy$

On the technical side, we are assuming here that $f'(x)$ is a continuous function (so that we're sure the integrals in the formulas exist – but that find of issue is more a topic for an advanced calculus course.)

Most of the examples for arc length that you can actually compute example are very "contrived" examples – because the formula for L often produces an integral that is very hard, if not impossible, to work out exactly. This doesn't mean that the integral is

useless, however. Apart from certain theoretical uses, you can always write down the integral for any arc length and then use the Midpoint Rule, or some more sophisticated approximation rule, to <u>*approximate the value of the integral*</u>*. For the Midpoint Rule, just pick as large an n as you can tolerate working with, subdivide the interval $[a, b]$ into n equal parts of length $\Delta x = \frac{b-a}{n}$, and plug into the formula for M_n.*

We can verify that these formulas <u>do</u> work in the simplest case (a straight line segment) which we did in Q1 without any calculus at all. (*Of course, the formula from precalculus was used, in the textbook, to derive the integral formula that works for* $y = f(x)$).

Q2: Write a calculus formula (*overkill !*) that will find the length of the graph of $y = f(x) = 2x - 1$ for $1 \le x \le 3$.

A) $\int_1^3 \sqrt{1 + (2x-1)^2}\, dx$ B) $\int_1^3 \sqrt{4}\, dx$ **C)** $\boldsymbol{\int_1^3 \sqrt{5}\, dx}$

D) $\int_1^3 \sqrt{(-1+2x)^2}\, dx$ E) $\int_1^3 3x\, dx$

<u>Answer</u> $\frac{dy}{dx} = \frac{d}{dx}(2x-1) = 2$,
so $L = \int_1^3 \sqrt{1 + (\frac{dy}{dx})^2}\, dx = \int_1^3 \sqrt{1 + (2)^2}\, dx = \int_1^3 \sqrt{5}\, dx$ ($= 2\sqrt{5}$, as above, using precalculus distance formula).

Q3: Write a calculus formula (*overkill !*) that will find the length of the graph of $x = \frac{1}{2}y + \frac{1}{2}$, $1 \le y \le 5$.

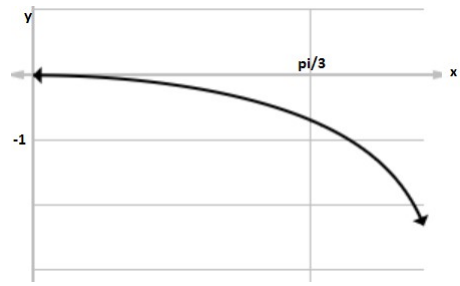
A) $\int_1^5 \sqrt{1 + (\frac{1}{2}y + \frac{1}{2})^2}\, dy$ B) $\int_1^5 \sqrt{\frac{1}{4}}\, dy$ C) $\int_1^5 \sqrt{5}\, dy$

D) $\int_1^5 \sqrt{(\frac{1}{2}y + \frac{1}{2})^2}\, dy$ E) $\int_1^5 \sqrt{\frac{5}{4}}\, dy$

This is exactly the same line segment: the only change has been to write $x = g(y)$ instead of $y = f(x)$.

Here $\frac{dx}{dy} = \frac{1}{2}$, so $L = \int_c^d \sqrt{1 + (\frac{dx}{dy})^2}\, dy = \int_1^5 \sqrt{1 + (\frac{1}{2})^2}\, dy = \int_1^5 \frac{\sqrt{5}}{2} dy = 2\sqrt{5}$.

<u>Example</u> Find the length of $y = f(x) = \ln(\cos x),\ 0 \le x \le \frac{\pi}{3}$.



$$f'(x) = \frac{dy}{dx} = \frac{1}{\cos x}(-\sin x) = -\tan x$$

$$1 + (\frac{dy}{dx})^2 = 1 + (-\tan x)^2 = \sec^2 x$$

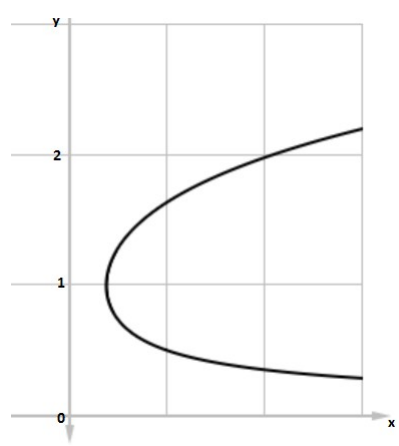
$$L = \int_0^{\pi/3} \sqrt{1 + (\frac{dy}{dx})^2}\, dx = \int_0^{\pi/3} \sqrt{\sec^2 x}\, dx = \int_0^{\pi/3} \sec x\, dx$$

$\uparrow$

(*because* $\sec^2 x \ge 0$ *for* $0 \le x \le \frac{\pi}{3}$)

$$= \ln|\sec x + \tan x| \Big|_0^{\pi/3} = \ln(2 + \sqrt{3})\ (\approx 1.3170)$$

<u>Example</u> Find the length of the graph of $x = g(x) = \frac{y^4}{8} + \frac{1}{4y^2} \qquad (1 \le y \le 2)$



$$g'(y) = \frac{dx}{dy} = \frac{1}{2}y^3 - \frac{1}{2y^3}$$

$$1 + (\frac{dx}{dy})^2 = 1 + (\frac{1}{2}y^3 - \frac{1}{2y^3})^2 = 1 + (\frac{1}{4}y^6 - \frac{1}{2} + \frac{1}{4y^6})$$

$$= \tfrac{1}{4}y^6 + \tfrac{1}{2} + \tfrac{1}{4y^6} = (\tfrac{1}{2}y^3 + \tfrac{1}{2y^3})^2$$

so $L \quad = \int_1^2 \sqrt{1 + (\frac{dx}{dy})^2}\, dy = \int_1^2 \sqrt{(\frac{1}{2}y^3 + \frac{1}{2y^3})^2}\, dy$

$$= \int_1^2 \tfrac{1}{2}y^3 + \tfrac{1}{2y^3}\, dy = (\tfrac{y^4}{8} - \tfrac{1}{4y^2})\Big|_1^2 = (2 - \tfrac{1}{16}) - (\tfrac{1}{8} - \tfrac{1}{4}) = \tfrac{31}{16} + \tfrac{2}{16} = \tfrac{33}{16}$$

<u>Example</u> Write the integral that represents the length of the graph of $y = \sin x \quad (0 \le x \le 2)$

$L = \int_0^2 \sqrt{1 + \cos^2 x}\; dx$. This relatively simple looking integral has no antiderivative that can be written in terms of elementary functions. We can only use some method (like M_n, for example) to get an approximate value for the integral. For example, $\mathrm{M}_{20} \approx 2.3515$ (*rounded to 4 decimal places. A <u>more accurate</u> approximate, rounded to 4 decimal places, in* 2.3517 – *obtained with a more accurate approximation rule called Simpson's Rule, also with* $n = 10$ *subdivisions).*