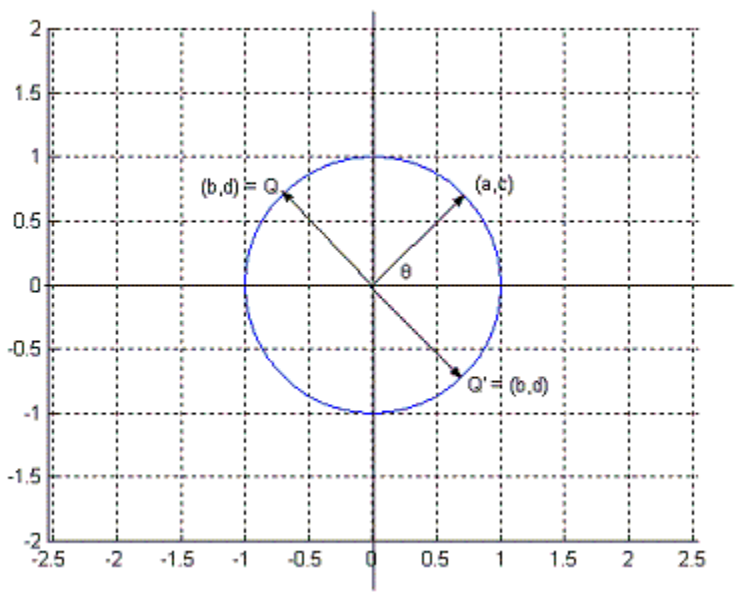
Orthogonal 2×2 Matrices

Suppose P is an $n \times n$ orthogonal matrix. Since P is square and $P^{-1} = P^T$, we have $1 = \det(PP^{-1}) = \det(PP^T) = (\det P)(\det P^T) = (\det P)^2$, so $\det P = \pm 1$.

Theorem Suppose $P = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is orthogonal. If

- if $\det P = 1$, then the mapping $\boldsymbol{x} \to P\boldsymbol{x}$ is a rotation
- if $\det P = -1$, then the mapping $\boldsymbol{x} \to P\boldsymbol{x}$ is a reflection across a line L through the origin (L is one of the eigenspaces of P)

Proof Since P is orthogonal, the columns are <u>orthonormal vectors</u> in $\mathbb{R}^2$, so $a^2 + c^2 = 1 = b^2 + d^2$. Therefore $\begin{bmatrix} a \\ c \end{bmatrix}$ and $\begin{bmatrix} b \\ d \end{bmatrix}$ are on the unit circle. Write $\begin{bmatrix} a \\ c \end{bmatrix} = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$, where θ is the angle between the positive x-axis and $\begin{bmatrix} a \\ c \end{bmatrix}$. Because $\begin{bmatrix} b \\ d \end{bmatrix}$ and $\begin{bmatrix} a \\ c \end{bmatrix}$ are orthogonal, $\begin{bmatrix} b \\ d \end{bmatrix}$ must be in one of the two positions Q or Q' (*see picture*); therefore $\begin{bmatrix} b \\ d \end{bmatrix} = \begin{bmatrix} \cos(\theta + \frac{\pi}{2}) \\ \sin(\theta + \frac{\pi}{2}) \end{bmatrix}$ or $\begin{bmatrix} b \\ d \end{bmatrix} = \begin{bmatrix} \cos(\theta + \frac{3\pi}{2}) \\ \sin(\theta + \frac{3\pi}{2}) \end{bmatrix}$.



So $P = \begin{bmatrix} \cos\theta & \cos\mu \\ \sin\theta & \sin\mu \end{bmatrix}$ where either $\mu = \theta + \frac{\pi}{2}$ or $\mu = \theta + \frac{3\pi}{2}$.

i) **If** $\boldsymbol{\mu = \theta + \frac{\pi}{2}}$, then

$$\begin{aligned} \cos\mu &= \cos\theta\cos\tfrac{\pi}{2} - \sin\theta\sin\tfrac{\pi}{2} &&= -\sin\theta \\ \sin\mu &= \sin\theta\cos\tfrac{\pi}{2} + \cos\theta\sin\tfrac{\pi}{2} &&= \cos\theta, \end{aligned}$$

so $P = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$, which is a <u>rotation</u> matrix.

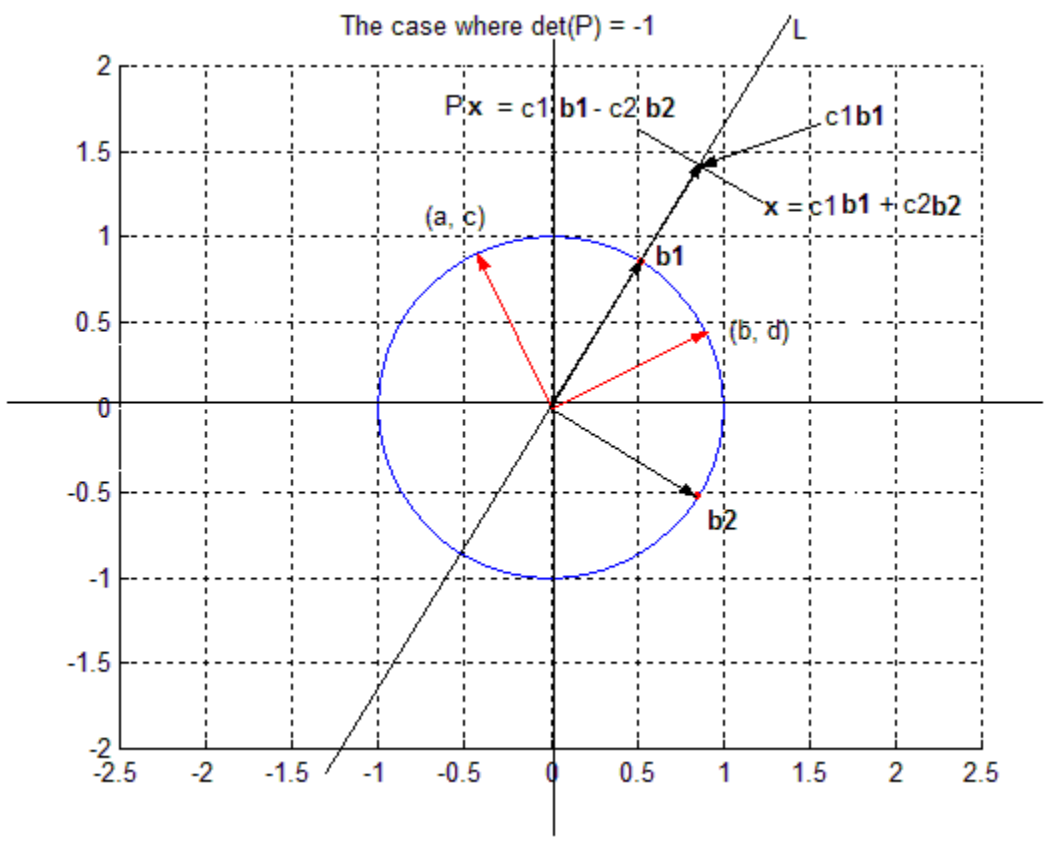
ii) **If $\boldsymbol{\mu = \theta + \frac{3\pi}{2}}$**, the same trig identities give $P = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}$. In this case, $\det\begin{bmatrix} \cos\theta - \lambda & \sin\theta \\ \sin\theta & -\cos\theta - \lambda \end{bmatrix} = \lambda^2 - 1$, so P has eigenvalues $\lambda_1 = 1$ and $\lambda_2 = -1$. Pick corresponding eigenvectors $\boldsymbol{b_1}$ and $\boldsymbol{b_2}$.

$$\begin{aligned} \text{Then } \boldsymbol{b_1} \cdot \boldsymbol{b_2} &= P\boldsymbol{b_1} \cdot P\boldsymbol{b_2} \qquad (\underline{\text{because}}\ P \text{ is orthogonal}) \\ &= (1)\,\boldsymbol{b_1} \cdot (-1)\boldsymbol{b_2} \\ &= -\boldsymbol{b_1} \cdot \boldsymbol{b_2}, \quad \text{so } \boldsymbol{b_1} \cdot \boldsymbol{b_2} = 0. \end{aligned}$$

Since $\boldsymbol{b_1}$ and $\boldsymbol{b_2}$ are orthogonal, they are linearly independent. (*Another reason: they are linearly independent because they are eigenvectors corresponding to different eigenvectors.*)

Therefore $\{\boldsymbol{b_1}, \boldsymbol{b_2}\}$ is an eigenvector basis for $\mathbb{R}^2$. So any $\boldsymbol{x}$ in $\mathbb{R}^2$ can be written as

$$\boldsymbol{x} = \boldsymbol{c_1 b_1} + \boldsymbol{c_2 b_2},$$

$$\text{so} \qquad P\boldsymbol{x} = c_1(1)\boldsymbol{b_1} + c_2(-1)\boldsymbol{b_2} = c_1\boldsymbol{b_1} - c_2\boldsymbol{b_2}.$$

This formula shows that the mapping $\boldsymbol{x} \mapsto P\boldsymbol{x}$ is a reflection across the line (eigenspace) $L = \text{Span}\{\boldsymbol{b_1}\}$ (*see the figure below).*