The Essential Norm of Operators on the Bergman Space

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This talk is based on joint work with:

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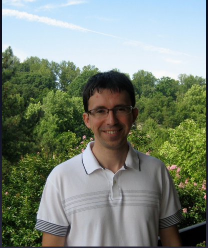

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Weighted Bergman Spaces on $\mathbb{B}_n$

- Let $\mathbb{B}_n := \{z \in \mathbb{C}^n : |z| < 1\}$.
- For $\alpha > -1$, we let

$$dv_\alpha(z) := c_\alpha (1-|z|^2)^\alpha \, dv(z), \quad \text{with} \quad c_\alpha := \frac{\Gamma(n+\alpha+1)}{n!\,\Gamma(\alpha+1)}.$$

  The choice of c_α gives that $v_\alpha(\mathbb{B}_n) = 1$.
- For $1 < p < \infty$ the space A^p_α is the collection of holomorphic functions on $\mathbb{B}_n$ such that

$$\|f\|^p_{A^p_\alpha} := \int_{\mathbb{B}_n} |f(z)|^p \, dv_\alpha(z) < \infty.$$

- For $\lambda \in \mathbb{B}_n$ let $k^{(p,\alpha)}_\lambda(z) = \frac{(1-|\lambda|^2)^{\frac{n+1+\alpha}{q}}}{(1-\overline{\lambda}z)^{n+1+\alpha}}$ and $K_\lambda(z) = \frac{1}{(1-\overline{\lambda}z)^{n+1+\alpha}}$.
- A computation shows: $\left\| k^{(p,\alpha)}_\lambda \right\|_{A^p_\alpha} \approx 1$.

Weighted Bergman Spaces on $\mathbb{D}^n$

- Let $\mathbb{D}^n := \{z \in \mathbb{C}^n : |z_l| < 1 \quad l = 1, \dots, n\}$.
- We let

$$dv_\alpha(z) := \frac{1}{\pi^n} dA(z_1) \cdots dA(z_n)$$

- For $1 < p < \infty$ the space A^p is the collection of holomorphic functions on $\mathbb{D}^n$ such that

$$\|f\|_{A^p(\mathbb{D}^n)}^p := \int_{\mathbb{D}^n} |f(z)|^p \, dv(z) < \infty.$$

- For $\lambda \in \mathbb{D}^n$ let $k_\lambda^{(p)}(z) := \prod_{l=1}^n \frac{(1-|\lambda_l|^2)^{\frac{2}{q}}}{(1-\overline{\lambda}_l z_l)^2}$ and $K_\lambda(z) = \prod_{l=1}^n \frac{1}{(1-\overline{\lambda}_l z_l)^2}$.
- A computation shows: $\left\| k_\lambda^{(p)} \right\|_{A^p(\mathbb{D}^n)} \approx 1$.

Toeplitz Operators and the Toeplitz Algebra

- The projection of L^2 onto A^2 is given by the integral operator

$$P(f)(z) := \int_{\Omega} f(w) K_w(z)\, dv(w).$$

- This operator is bounded from L^p to A^p when $1 < p < \infty$.
- Let M_a denote the operator of multiplication by the function a, $M_a(f) := af$. The Toeplitz operator with symbol $a \in L^\infty$ is the operator given by

$$T_a := PM_a.$$

- It is immediate to see that $\|T_a\|_{\mathcal{L}(A^p)} \lesssim \|a\|_{L^\infty}$.
- More generally, for a measure μ we will define the operator

$$T_\mu f(z) := \int_{\Omega} f(w) K_w(z)\, d\mu(w),$$

which will define an analytic function for all $f \in H^\infty$.

Toeplitz Operators and the Toeplitz Algebra

- For symbols in L^∞ we let $\mathcal{T}_p$ be the C^* subalgebra of $\mathcal{L}(A^p)$ generated by T_a.
- An important class of operators in $\mathcal{T}_p$ are those that are finite sums of finite products of Toeplitz operators.
  Namely, for symbols $a_{jk} \in L^\infty$ with $1 \leq j \leq J$ and $1 \leq k \leq K$ we will need to study the operators:

$$\sum_{j=1}^{J} \prod_{k=1}^{K} T_{a_{jk}}$$

- Additionally,

$$\mathcal{T}_p = \overline{\left\{ \sum_{j=1}^{J} \prod_{k=1}^{K} T_{a_{jk}} : a_{jk} \in L^\infty \quad 1 \leq j \leq J \quad 1 \leq k \leq K \right\}}^{\mathcal{L}(A^p)}$$

Geometry of the Domain Ω

When $\Omega = \mathbb{B}_n$ let

$$d\lambda(z) := \frac{dv(z)}{(1-|z|^2)^{n+1}}$$

and when $\Omega = \mathbb{D}^n$ let

$$d\lambda(z) := \frac{dv(z)}{\prod_{l=1}^{n}(1-|z_l|^2)^2}$$

Lemma (Geometric Decomposition of Ω)

There exists an integer $N > 0$ (depending only on the doubling constant of the measure λ) such that for any $r > 0$ there is a covering $\mathcal{F}_r = \{F_j\}$ of Ω by disjoint Borel sets satisfying

(1) *every point of Ω belongs to at most N of the sets $G_j := \{z \in \Omega : d(z, F_j) \leq r\}$,*

(2) $\operatorname{diam}_\beta F_j \leq 2r$ *for every j.*

Carleson Measures for $A^p_\alpha(\mathbb{B}_n)$

A measure μ on $\mathbb{B}_n$ is a Carleson measure for A^p_α if

$$\int_{\mathbb{B}_n} |f(z)|^p \, d\mu(z) \lesssim \int_{\mathbb{B}_n} |f(z)|^p \, dv_\alpha(z) \quad \forall f \in A^p_\alpha.$$

Lemma (Characterizations of $A^p_\alpha(\mathbb{B}_n)$ Carleson Measures)

Suppose that $1 < p < \infty$ and $\alpha > -1$. Let μ be a measure on $\mathbb{B}_n$ and $r > 0$. The following quantities are equivalent, with constants that depend on n, α and r:

(1) $\|\mu\|_{\mathrm{RKM}} := \sup_{z\in\mathbb{B}_n} \int_{\mathbb{B}_n} \frac{(1-|z|^2)^{n+1+\alpha}}{|1-\overline{z}w|^{2(n+1+\alpha)}} \, d\mu(w)$;

(2) $\|\imath_p\|^p := \inf\left\{ C : \int_{\mathbb{B}_n} |f(z)|^p \, d\mu(z) \leq C \int_{\mathbb{B}_n} |f(z)|^p \, dv_\alpha(z) \right\}$;

(3) $\|\mu\|_{\mathrm{Geo}} := \sup_{z\in\mathbb{B}_n} \frac{\mu(D(z,r))}{\left(1-|z|^2\right)^{n+1+\alpha}}$;

(4) $\|T_\mu\|_{\mathcal{L}(A^p_\alpha(\mathbb{B}_n))}$.

Carleson Measures for $A^p(\mathbb{D}^n)$

A measure μ on $\mathbb{D}^n$ is a Carleson measure for A^p_α if

$$\int_{\mathbb{D}^n} |f(z)|^p \, d\mu(z) \lesssim \int_{\mathbb{D}^n} |f(z)|^p \, dv(z) \quad \forall f \in A^p.$$

Lemma (Characterizations of $A^p(\mathbb{D}^n)$ Carleson Measures)

Suppose that $1 < p < \infty$. Let μ be a measure on $\mathbb{D}^n$ and $r > 0$. The following quantities are equivalent, with constants that depend on n, α and r:

(1) $\|\mu\|_{\mathrm{RKM}} := \sup_{z\in\mathbb{D}^n} \int_{\mathbb{D}^n} \prod_{l=1}^n \frac{(1-|z_l|^2)^2}{|1-\overline{z}_l w_l|^4} \, d\mu(w)$;

(2) $\|\imath_p\|^p := \inf\left\{C : \int_{\mathbb{D}^n} |f(z)|^p \, d\mu(z) \leq C \int_{\mathbb{D}^n} |f(z)|^p \, dv(z)\right\}$;

(3) $\|\mu\|_{\mathrm{Geo}} = \sup_{z\in\mathbb{D}^n} \frac{\mu(D(z,r))}{\prod_{l=1}^n \left(1-|z_l|^2\right)^2} \approx \sup_{z\in\mathbb{D}^n} \frac{\mu(D(z,r))}{v(D(z,r))}$;

(4) $\|T_\mu\|_{\mathcal{L}(A^p(\mathbb{D}^n))}$.

The Berezin Transform

For $S \in \mathcal{L}(A^p)$, we define the Berezin transform by

$$B(S)(z) := \left\langle Sk_z^{(p)}, k_z^{(q)} \right\rangle_{A^2}.$$

- $B : \mathcal{L}(A^p) \to L^\infty(\Omega)$.
- If S is compact, then $B(S)(z) \to 0$ as $z \to \partial\Omega$.
- The Berezin transform is one-to-one.
- $B(S)$ is Lipschitz conitnuous with respect to the hyperbolic metric

$$|B(S)(z_1) - B(S)(z_2)| \leq \sqrt{2} \, \|S\|_{\mathcal{L}(A^p)} \, \beta(z_1, z_2)$$

- Range of B is not closed: $B^{-1} : B(\mathcal{L}(A^p)) \to \mathcal{L}(A^p)$ is not bounded.

Related Results

Theorem (Axler and Zheng, Indiana Univ. Math. J. 47 (1998))

Suppose that $a_{jk} \in L^\infty(\mathbb{D})$ *with* $1 \leq j \leq J$ *and* $1 \leq k \leq K$. *Let* $S = \sum_{j=1}^{J} \prod_{k=1}^{K} T_{a_{jk}}$ *The following are equivalent:*

(a) *The operator* S *is compact on* $A^2(\mathbb{D})$;

(b) $B(S)(z) \to 0$ *as* $|z| \to 1$;

(c) $\|Sk_z\|_{A^2} \to 0$ *as* $|z| \to 1$.

- The interesting implication is $(b) \Rightarrow (a)$;
- The same proof works in the case of the unit ball, but was done by Raimondo. Also works in the polydisc by Nam and Zheng.

Theorem (Engliš, Ark. Mat. 30 (1992))

Let $1 < p < \infty$. *If* S *is a compact operator on* A^p, *then* $S \in \mathcal{T}_p$.

Main Question of Interest

From the previous Theorem and simple functional analysis we have that if S is compact on A^p then

$$S \in \mathcal{T}_p \quad \text{and} \quad B(S)(z) \to 0 \text{ as } z \to \partial\Omega.$$

Question (Characterizing the Compacts)

If $S \in \mathcal{T}_p$ and $B(S)(z) \to 0$ as $z \to \partial\Omega$, then is S is compact?

Yes!

- Shown to be true by Suárez for A^p when $1 < p < \infty$ and $\alpha = 0$.
- Extended to $\alpha > -1$ by Suárez, Mitkovski and BDW.
- Extended to the polydisc $\mathbb{D}^n$ by Mitkovski and BDW.

Characterizations of Compactness and Essential Norm

Theorem

Let $1 < p < \infty$ and $S \in \mathcal{L}(A^p)$. Then S is compact if and only if $S \in \mathcal{T}_p$ and $\lim_{z\to\partial\Omega} B(S)(z) = 0$.

We can actually obtain much more precise information about the essential norm of an operator. For $S \in \mathcal{L}(A^p_\alpha)$ recall that

$$\|S\|_e = \inf\left\{\|S - Q\|_{\mathcal{L}(A^p_\alpha)} : Q \text{ is compact}\right\}.$$

Two other measures of the "size" of an operator $S \in \mathcal{L}(A^p)$:

$$\begin{aligned} \mathfrak{b}_S &:= \sup_{r>0} \limsup_{z\to\partial\Omega} \left\| M_{1_{D(z,r)}} S \right\|_{\mathcal{L}(A^p,L^p)} \\ \mathfrak{c}_S &:= \lim_{r\to 1} \left\| M_{1_{\Omega\setminus r\Omega}} S \right\|_{\mathcal{L}(A^p,L^p)}. \end{aligned}$$

Characterizations of Compactness and Essential Norm

Let $r > 0$ and let $\{w_m\}$ and D_m be the sets that form a lattice in Ω. Define the measure

$$\mu_r = \sum_m v(D_m)\delta_{w_m}.$$

It is well know that μ_r is a A^p Carleson measure, so $T_{\mu_r} : A^p \to A^p$ is bounded.

Lemma

$T_{\mu_r} \to Id$ on $\mathcal{L}(A^p)$ when $r \to 0$.

Let $r > 0$ be chosen so that $\| T_{\mu_r} - Id\|_{\mathcal{L}(A^p)} < \frac{1}{4}$, and $\mu := \mu_r$. Then set

$$\mathfrak{a}_S(\rho) := \limsup_{z\to\partial\Omega} \sup \left\{ \|Sf\|_{A^p} : f \in T_{\mu 1_{D(z,\rho)}}(A^p),\ \|f\|_{A^p} \leq 1 \right\}$$

and define

$$\mathfrak{a}_S := \lim_{\rho\to 1} \mathfrak{a}_S(\rho).$$

Characterizations of Compactness and Essential Norm

Theorem

Let $1 < p < \infty$ and let $S \in \mathcal{T}_p$. Then there exists constants depending only on n and p such that:

$$\mathfrak{a}_S \approx \mathfrak{b}_S \approx \mathfrak{c}_S \approx \|S\|_e .$$

For the automorphism φ_z such that $\varphi_z(0) = z$ define the map

$$U_z^{(p)} f(w) := f(\varphi_z(w)) \frac{(1 - |z|^2)^{\frac{n+1+\alpha}{p}}}{(1 - w\overline{z})^{\frac{2(n+1+\alpha)}{p}}}.$$

A standard change of variable argument and computation gives that

$$\left\| U_z^{(p)} f \right\|_{A^p} = \|f\|_{A^p} \quad \forall f \in A^p.$$

Characterizations of Compactness and Essential Norm

For $z \in \Omega$ and $S \in \mathcal{L}(A^p)$ we then define the map

$$S_z := U_z^{(p,\alpha)} S (U_z^{(q,\alpha)})^*.$$

One should think of the map S_z in the following way. This is an operator on A^p and so it first acts as "translation" in Ω, then the action of S, then "translation" back.

Theorem

Let $1 < p < \infty$ *and* $S \in \mathcal{T}_p$*. Then*

$$\|S\|_e \approx \sup_{\|f\|_{A^p}=1} \limsup_{z \to \partial\Omega} \|S_z f\|_{A^p}.$$

Connecting the Geometry and Operator Theory

Lemma

Let $S \in \mathcal{T}_p$, μ *a Carleson measure and* $\epsilon > 0$. *Then there are Borel sets* $F_j \subset G_j \subset \Omega$ *such that*

(i) $\Omega = \cup F_j$;

(ii) $F_j \cap F_k = \emptyset$ *if* $j \neq k$;

(iii) *each point of* Ω *lies in no more than* $N(n)$ *of the sets* G_j;

(iv) $\operatorname{diam}_\beta G_j \leq d(p, S, \epsilon)$

and

$$\left\| ST_\mu - \sum_{j=1}^{\infty} M_{1_{F_j}} S T_{1_{G_j}\mu} \right\|_{\mathcal{L}(A^p, L^p)} < \epsilon.$$

A Uniform Algebra and its Maximal Ideal Space

- Let $\mathcal{A}$ denote the bounded functions that are uniformly continuous from the metric space (Ω, β) into the metric space $(\mathbb{C}, |\cdot|)$.
- Associate to $\mathcal{A}$ its maximal ideal space $M_{\mathcal{A}}$ which is the set of all non-zero multiplicative linear functionals from $\mathcal{A}$ to $\mathbb{C}$.
- Since $\mathcal{A}$ is a C^* algebra we have that Ω is dense in $M_{\mathcal{A}}$.
- The Toeplitz operators associated to symbols in $\mathcal{A}$ are useful to study the Toeplitz algebra $\mathcal{T}_p$.

Theorem

The Toeplitz algebra $\mathcal{T}_p$ equals the closed algebra generated by $\{T_a : a \in \mathcal{A}\}$.

A Uniform Algebra and its Maximal Ideal Space

- For an element $x \in M_{\mathcal{A}} \setminus \Omega$ choose a net $z_\omega \to x$.
- Form S_{z_ω} and look at the limit operator obtained when $z_\omega \to x$, denote it by S_x.

Lemma

Let $S \in \mathcal{L}(A^p)$. Then $B(S)(z) \to 0$ as $z \to \partial\Omega$ if and only if $S_x = 0$ for all $x \in M_{\mathcal{A}} \setminus \Omega$.

We can extend this to compute the essential norm of an operator S in terms of S_x where $x \in M_{\mathcal{A}} \setminus \Omega$.

Theorem

Let $S \in \mathcal{T}_p$. Then there exists a constant $C(p,n)$ such that

$$\sup_{x \in M_{\mathcal{A}} \setminus \Omega} \|S_x\|_{\mathcal{L}(A^p)} \approx \|S\|_e .$$

Proof of Main Theorem

Theorem

Let $1 < p < \infty$ *and* $S \in \mathcal{L}(A^p)$*. Then* S *is compact if and only if* $S \in \mathcal{T}_p$ *and* $\lim_{z\to\partial\Omega} B(S)(z) = 0$*.*

Proof.

$\Rightarrow$: If S is compact that $B(S)(z) \to 0$ as $z \to \partial\Omega$ and $S \in \mathcal{T}_p$.
$\Leftarrow$: If $S \in \mathcal{T}_p$, then we have

$$\sup_{x\in M_{\mathcal{A}}\setminus\Omega} \|S_x\|_{\mathcal{L}(A^p)} \approx \|S\|_e .$$

If $B(S)(z) \to 0$ as $z \to \partial\Omega$, then $S_x = 0$ for all $x \in M_{\mathcal{A}} \setminus \Omega$.
This gives $\|S\|_e = 0$ or equivalently S is compact. □

Thank You!