Carleson Measures for Hilbert Spaces of Analytic Functions

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Setup and General Overview

Let Ω be an open set in $\mathbb{C}^n$;
Let $\mathcal{H}$ be a Hilbert function space over Ω with reproducing kernel K_λ:

$$f(\lambda) = \langle f, K_\lambda \rangle_{\mathcal{H}}.$$

Definition ($\mathcal{H}$-Carleson Measure)

A non-negative measure μ on Ω is $\mathcal{H}$-Carleson if and only if

$$\int_\Omega |f(z)|^2 \, d\mu(z) \leq C(\mu)^2 \, \|f\|_{\mathcal{H}}^2 .$$

Question

Give a 'geometric' and 'testable' characterization of the $\mathcal{H}$-Carleson measures.

Obvious Necessary Conditions for Carleson Measures

Let k_λ denote the normalized reproducing kernel for the space $\mathcal{H}$:

$$k_\lambda(z) = \frac{K_\lambda(z)}{\|K_\lambda\|_{\mathcal{H}}}.$$

Testing on the reproducing kernel k_λ we always have a necessary geometric condition for the measure μ to be Carleson:

$$\sup_{\lambda \in \Omega} \int_\Omega |k_\lambda(z)|^2 \, d\mu(z) \leq C(\mu)^2.$$

In the cases of interest it is possible to identify a point $\lambda \in \Omega$ with an open set, I_λ on the boundary of Ω. A 'geometric' necessary condition is:

$$\mu\left(T(I_\lambda)\right) \lesssim \|K_\lambda\|_{\mathcal{H}}^{-2}.$$

Here $T(I_\lambda)$ is the 'tent' over the set I_λ in the boundary $\partial\Omega$.

Reasons to Care about Carleson Measures

- Bessel Sequences/Interpolating Sequences/Riesz Sequences: Given $\Lambda = \{\lambda_j\}_{j=1}^{\infty} \subset \Omega$ determine functional analytic basis properties for the set $\{k_{\lambda_j}\}_{j=1}^{\infty}$:

  $\{k_{\lambda_j}\}_{j=1}^{\infty}$ is a Bessel sequence if and only if μ_Λ is $\mathcal{H}$-Carleson;
  $\{k_{\lambda_j}\}_{j=1}^{\infty}$ is a Riesz sequence if and only if μ_Λ is $\mathcal{H}$-Carleson and separated.

- Multipliers of $\mathcal{H}$: Characterize the pointwise multipliers for $\mathcal{H}$:

$$\begin{aligned} \text{Multi}(\mathcal{H}) &= H^\infty \cap CM(\mathcal{H}). \\ \|\varphi\|_{\text{Multi}(\mathcal{H})} &\approx \|\varphi\|_{H^\infty} + \|\mu_\varphi\|_{CM(\mathcal{H})}. \end{aligned}$$

- Commutator/Bilinear Form/Hankel Form/Paraproduct Estimates: Given b, define $T_b : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$ by:

$$\begin{aligned} T_b(f,g) &= \langle fg, b\rangle_{\mathcal{H}}. \\ \|T_b\|_{\mathcal{H}\times\mathcal{H}\to\mathbb{C}} &\approx \|\mu_b\|_{CM(\mathcal{H})}. \end{aligned}$$

Choose your own talk?

While at a conference you discover that you are interested in a certain class of Carleson measures for a Hilbert space of analytic functions. While looking into this question, you come to a fork in the road and must choose which direction to proceed. Each direction has challenges, but miraculously in both directions the challenges can be overcome with similar tools! Which way do you choose....

- I am an analyst that cares more about several complex variables, function theory, Carleson measures, and their interaction.

  ▸ Characterization of Carleson Measures for Besov-Sobolev Space $B_2^\sigma(\mathbb{B}_n)$

- I am an analyst that cares mores about one complex variable, inner functions, Carleson measures, and their interaction.

  ▸ Characterization of Carleson Measures for the Model Space K_ϑ on $\mathbb{D}$

Besov-Sobolev Spaces

- The space $B_2^\sigma(\mathbb{B}_n)$ is the collection of holomorphic functions f on the unit ball $\mathbb{B}_n = \{z \in \mathbb{C}^n : |z| < 1\}$ such that

$$\left\{ \sum_{k=0}^{m-1} \left| f^{(k)}(0) \right|^2 + \int_{\mathbb{B}_n} \left| \left(1 - |z|^2\right)^{m+\sigma} f^{(m)}(z) \right|^2 d\lambda_n(z) \right\}^{\frac{1}{2}} < \infty,$$

  where $d\lambda_n(z) = \left(1 - |z|^2\right)^{-n-1} dV(z)$ is the invariant measure on $\mathbb{B}_n$ and $m + \sigma > \frac{n}{2}$.
- Various choices of σ recover important classical function spaces:
  - $\sigma = 0$: Corresponds to the Dirichlet Space;
  - $\sigma = \frac{1}{2}$: Drury-Arveson Hardy Space;
  - $\sigma = \frac{n}{2}$: Classical Hardy Space;
  - $\sigma > \frac{n}{2}$: Bergman Spaces.

Besov-Sobolev Spaces

- The spaces $B_2^\sigma(\mathbb{B}_n)$ are reproducing kernel Hilbert spaces:

$$\forall\lambda \in \mathbb{B}_n \quad f(\lambda) = \langle f, K_\lambda^\sigma \rangle_{B_2^\sigma(\mathbb{B}_n)} \quad \forall f \in B_2^\sigma(\mathbb{B}_n).$$

- A computation shows the kernel function $K_\lambda^\sigma(z)$ is:

$$K_\lambda^\sigma(z) = \frac{1}{\left(1 - \overline{\lambda} z\right)^{2\sigma}}$$

  - $\sigma = \frac{1}{2}$: Drury-Arveson Hardy Space; $K_\lambda^{\frac{1}{2}}(z) = \frac{1}{1-\overline{\lambda}z}$;
  - $\sigma = \frac{n}{2}$: Classical Hardy Space; $K_\lambda^{\frac{n}{2}}(z) = \frac{1}{(1-\overline{\lambda}z)^n}$;
  - $\sigma = \frac{n+1}{2}$: Bergman Space; $K_\lambda^{\frac{n+1}{2}}(z) = \frac{1}{(1-\overline{\lambda}z)^{n+1}}$.

Carleson Measures for Besov-Sobolev Spaces

We always have the following necessary condition:

$$\mu(T(B_r)) \lesssim r^{2\sigma}.$$

When $0 \leq \sigma \leq \frac{1}{2}$:

- If $n = 1$, the characterization can be expressed in terms of capacitary conditions. More precisely,

$$\mu(T(G)) \lesssim \operatorname{cap}_\sigma(G) \quad \forall \text{open } G \subset \mathbb{T}.$$

  See for example Stegenga, Maz'ya, Verbitsky, Carleson.
- If $n > 1$ then there are different characterizations of Carleson measures for $B_2^\sigma(\mathbb{B}_n)$:
  - Capacity methods of Cohn and Verbitsky.
  - Dyadic tree structures on the ball by Arcozzi, Rochberg, and Sawyer.
  - Testing Conditions on indicators ("T(1)" conditions) by Tchoundja.

Question (Main Problem: Characterization in the Difficult Range)

Characterize the Carleson measures when $\frac{1}{2} < \sigma < \frac{n}{2}$.

Operator Theoretic Characterization of Carleson Measures

A measure μ is Carleson exactly if the inclusion map ι from $\mathcal{H}$ to $L^2(\Omega;\mu)$ is bounded, or

$$\int_\Omega |f(z)|^2 \, d\mu(z) \leq C(\mu)^2 \, \|f\|_{\mathcal{H}}^2 .$$

A simple functional analysis argument lets one recast this in an equivalent way:

Proposition (Arcozzi, Rochberg, and Sawyer)

A measure μ is a $\mathcal{H}$-Carleson measure if and only if the linear map

$$T(f)(z) = \int_\Omega \operatorname{Re} K_x(z)\, f(x) d\mu(x)$$

is bounded on $L^2(\Omega;\mu)$.

Connections to Calderón-Zygmund Operators

When we apply this proposition to the spaces $B_2^\sigma(\mathbb{B}_n)$ this suggests that we study the operator

$$T_{\mu,2\sigma}(f)(z) = \int_{\mathbb{B}_n} \mathrm{Re}\left(\frac{1}{(1-\overline{w}z)^{2\sigma}}\right) f(w)d\mu(w) : L^2(\mathbb{B}_n;\mu) \to L^2(\mathbb{B}_n;\mu)$$

and find some conditions that will let us determine when it is bounded.

- The kernel of the above integral operator has some cancellation and size estimates that are reminiscent of Calderón-Zygmund operators as living on a smaller dimensional space.
- The measure μ has a growth condition similar to the estimates on the kernel.
- Idea: Try to use the T(1)-Theorem from harmonic analysis to characterize the boundedness of

$$T_{\mu,2\sigma} : L^2(\mathbb{B}_n;\mu) \to L^2(\mathbb{B}_n;\mu).$$

Danger: Proof will Fail without Coordination!

Calderón–Zygmund Estimates for $T_{\mu,2\sigma}$

If we define

$$\Delta(z,w) := \begin{cases} \left||z|-|w|\right| + \left|1-\frac{z\overline{w}}{|z||w|}\right| & : \quad z,w\in\mathbb{B}_n\setminus\{0\} \\ |z|+|w| & : \quad \text{otherwise.} \end{cases}$$

Then Δ is a pseudo-metric and makes the ball into a space of homogeneous type.

A computation demonstrates that the kernel of $T_{\mu,2\sigma}$ satisfies the following estimates:

$$|K_{2\sigma}(z,w)| \lesssim \frac{1}{\Delta(z,w)^{2\sigma}} \quad \forall z,w\in\mathbb{B}_n;$$

If $\Delta(\zeta,w) < \frac{1}{2}\Delta(z,w)$ then

$$|K_{2\sigma}(\zeta,w) - K_{2\sigma}(z,w)| \lesssim \frac{\Delta(\zeta,w)^{1/2}}{\Delta(z,w)^{2\sigma+1/2}}.$$

Calderón-Zygmund Estimates for $T_{\mu,2\sigma}$

- These estimates on $K_{2\sigma}(z, w)$ say that it is a Calderón-Zygmund kernel of order 2σ with respect to the metric Δ.
  - Unfortunately, we can't apply the standard T(1) technology (adapted to a space of homogeneous type) to study the operators $T_{\mu,2\sigma}$. We would need the estimates of order n instead of 2σ.
- However, the measures we want to study (the Carleson measures for the space) satisfy the growth estimate

  $$\mu\left(T(B_r)\right) \lesssim r^{2\sigma}$$

  and this is exactly the phenomenon that will save us!
- This places us in the setting of non-homogeneous harmonic analysis as developed by Nazarov, Treil, and Volberg. We have an operator with a Calderón-Zygmund kernel satisfying estimates of order 2σ, a measure μ of order 2σ, and are interested in $L^2(\mathbb{B}_n; \mu) \to L^2(\mathbb{B}_n; \mu)$ bounds.

Euclidean Variant of the Question

Their is a natural extension of these questions/ideas to the Euclidean setting $\mathbb{R}^d$.

More precisely, for $m \leq d$ we are interested in Calderón-Zygmund kernels that satisfy the following estimates:

$$|k(x,y)| \leq \frac{C_{CZ}}{|x-y|^m},$$

and

$$|k(y,x) - k(y,x')| + |k(x,y) - k(x',y)| \leq C_{CZ} \frac{|x-x'|^\tau}{|x-y|^{m+\tau}}$$

provided that $|x-x'| \leq \frac{1}{2}|x-y|$, with some (fixed) $0 < \tau \leq 1$ and $0 < C_{CZ} < \infty$.

Euclidean Variant of the Question

Additionally the kernels will have the following property

$$|k(x,y)| \leq \frac{1}{\max(d(x)^m, d(y)^m)},$$

where $d(x) := \mathrm{dist}(x, \mathbb{R}^d \setminus H)$ and H being an open set in $\mathbb{R}^d$.
Key examples: Let $H = \mathbb{B}_d$, the unit ball in $\mathbb{R}^d$ and

$$k(x,y) = \frac{1}{(1 - x \cdot y)^m}.$$

We will say that k is a Calderón-Zygmund kernel on a closed $X \subset \mathbb{R}^d$ if $k(x,y)$ is defined only on $X \times X$ and the previous properties of k are satisfied whenever $x, x', y \in X$.
Once the kernel has been defined, then we say that a $L^2(\mathbb{R}^d; \mu)$ bounded operator is a Calderón-Zygmund operator with kernel k if,

$$T_{\mu,m} f(x) = \int_{\mathbb{R}^d} k(x,y) f(y) d\mu(y) \quad \forall x \notin \mathrm{supp} f\ .$$

T(1)-Theorem for Bergman-Type Operators

Theorem (T(1)-Theorem for Bergman-Type Operators, Volberg and W., Amer. J. Math., **134** (2012))

Let $k(x,y)$ be a Calderón-Zygmund kernel of order m on $X \subset \mathbb{R}^d$, $m \leq d$ with Calderón-Zygmund constants C_{CZ} and τ. Let μ be a probability measure with compact support in X and all balls such that $\mu(B_r(x)) > r^m$ lie in an open set H. Let also

$$|k(x,y)| \leq \frac{1}{\max(d(x)^m, d(y)^m)},$$

where $d(x) := \operatorname{dist}(x, \mathbb{R}^d \setminus H)$. Finally, suppose also that:

$$\| T_{\mu,m}\chi_Q\|_{L^2(\mathbb{R}^d;\mu)}^2 \leq A\,\mu(Q)\,,\ \| T_{\mu,m}^*\chi_Q\|_{L^2(\mathbb{R}^d;\mu)}^2 \leq A\,\mu(Q)\,.$$

Then $\| T_{\mu,m}\|_{L^2(\mathbb{R}^d;\mu)\to L^2(\mathbb{R}^d;\mu)} \leq C(A,m,d,\tau)$.

Main Results

Theorem (Characterization of Carleson Measures for $B^2_\sigma(\mathbb{B}_n)$, Volberg and W., Amer. J. Math., 134 (2012))

Let μ be a non-negative Borel measure in $\mathbb{B}_n$. The following conditions are equivalent:

(a) *μ is a $B_2^\sigma(\mathbb{B}_n)$-Carleson measure;*

(b) *$T_{\mu,2\sigma} : L^2(\mathbb{B}_n;\mu) \to L^2(\mathbb{B}_n;\mu)$ is bounded;*

(c) *There is a constant C such that*

  (i) $\|T_{\mu,2\sigma}\chi_Q\|^2_{L^2(\mathbb{B}_n;\mu)} \leq C\,\mu(Q)$ *for all Δ-cubes Q;*

  (ii) $\mu(B_\Delta(x,r)) \leq C\, r^{2\sigma}$ *for all balls $B_\Delta(x,r)$ that intersect $\mathbb{C}^n \setminus \mathbb{B}_n$.*

Above, the sets B_Δ are balls measured with respect to the metric Δ and the set Q is a "cube" defined with respect to the metric Δ.

Remarks about Characterization of Carleson Measures

- We have already seen that $(a) \Leftrightarrow (b)$, and it is trivial $(b) \Rightarrow (c)$.
- It only remains to prove that $(c) \Rightarrow (b)$.
  - The proof of this theorem follows from a real variable harmonic analysis proof of the T(1)-Theorem for Bergman-type operators.
  - Follow the proof strategy for the T(1) theorem in the context at hand. Technical but well established path (safe route!).
- It is possible to show that the T(1) condition reduces to the simpler conditions in certain cases.
- An alternate proof of this Theorem was later given by Hytönen and Martikainen. Their proof used a non-homogeneous T(b)-Theorem on metric spaces!

Safe Passage to the End!

Return to Beginning | Conclusion | Details

The Model Space

Let H^2 denote the Hardy space on the unit disc $\mathbb{D}$;
Let ϑ denote an inner function on $\mathbb{D}$:

$$|\vartheta(\xi)| = 1 \quad \text{a.e. } \xi \in \mathbb{T}.$$

Let $K_\vartheta = H^2 \ominus \vartheta H^2$.
This is a reproducing kernel Hilbert space with kernel:

$$K_\lambda(z) = \frac{1 - \overline{\vartheta(\lambda)}\vartheta(z)}{1 - \overline{\lambda} z}.$$

Question (Carleson Measure Problem for K_ϑ)

Geometrically characterize the Carleson measures for K_ϑ:

$$\int_{\overline{\mathbb{D}}} |f(z)|^2 \, d\mu(z) \leq C(\mu)^2 \, \|f\|_{K_\vartheta}^2 \quad \forall f \in K_\vartheta.$$

Carleson Measures for K_ϑ

We always have the necessary condition:

$$\int_{\mathbb{D}} \frac{\left|1-\overline{\vartheta(\lambda)}\vartheta(z)\right|^2}{\left|1-\overline{\lambda}z\right|^2} d\mu(z) \leq C(\mu)^2 \left\|K_\lambda\right\|_{K_\vartheta}^2 \quad \forall \lambda \in \mathbb{D}.$$

- If ϑ is a one-component inner function: Namely,

  $$\Omega(\epsilon) \equiv \{z \in \mathbb{D} : |\vartheta(z)| < \epsilon\}, \qquad 0 < \epsilon < 1$$

  is connected for some ϵ:
  - Cohn proved that μ is a K_ϑ-Carleson measure if and only if the testing conditions hold for Carleson boxes that intersect $\Omega(\epsilon)$.
  - Treil and Volberg gave an alternate proof of this. Their proof works for $1 < p < \infty$.
- Nazarov and Volberg proved the obvious necessary condition is not sufficient for μ to be a K_ϑ-Carleson measure.

The Two-Weight Cauchy Transform

- Let σ denote a measure on $\mathbb{R}$.
- Let τ denote a measure on $\overline{\mathbb{R}^2_+}$.
- For $f \in L^2(\mathbb{R}, \sigma)$, the Cauchy transform will be

$$\mathrm{C}_\sigma(f)(z) = \int_{\mathbb{R}} \frac{f(w)}{w - z} \sigma(dw) = \mathrm{C}(\sigma f)(z).$$

- Let σ denote a measure on $\mathbb{T}$.
- Let τ denote a measure on $\overline{\mathbb{D}}$.
- For $f \in L^2(\mathbb{T}, \sigma)$, the Cauchy transform will be

$$\mathrm{C}_\sigma(f)(z) = \int_{\mathbb{T}} \frac{f(w)}{1 - \overline{w} z} \sigma(dw) = \mathrm{C}(\sigma f)(z).$$

Connecting Carleson Measures to the Cauchy Transform

- Associate to ϑ and $\alpha \in \mathbb{T}$ a measure σ_α such that:

$$\text{Re}\left(\frac{\alpha + \vartheta(z)}{\alpha - \vartheta(z)}\right) = \int_{\mathbb{T}} \frac{1 - |z|^2}{\left|1 - \bar{\xi} z\right|^2}\, d\sigma_\alpha(\xi).$$

  Let $\sigma \equiv \sigma_1$ denote the Clark measure on $\mathbb{T}$.
- Then $L^2(\mathbb{T}; \sigma)$ is unitarily equivalent to K_ϑ via a unitary U.
- $U^* : L^2(\mathbb{T}; \sigma) \to K_\vartheta$ has the integral representation given by

$$U^* f(z) \equiv (1 - \vartheta(z)) \int_{\mathbb{T}} \frac{f(\xi)}{1 - \bar{\xi} z} \sigma(d\xi).$$

- For the inner function ϑ and measure μ, define a new measure $\nu_{\vartheta,\mu} \equiv |1 - \vartheta|^2 \mu$.

Lemma (Nazarov, Volberg)

A measure μ is a Carleson measure for K_ϑ if and only if $\mathrm{C} : L^2(\mathbb{T}; \sigma) \to L^2(\overline{\mathbb{D}}; \nu_{\vartheta,\mu})$ is bounded.

Two-Weight Inequality for the Cauchy Transform

Theorem (Lacey, Sawyer, Shen, Uriarte-Tuero, W.)

Let σ be a weight on $\mathbb{T}$ and τ a weight on $\overline{\mathbb{D}}$. The inequality below holds, for some finite positive $\mathscr{C}$,

$$\|\mathrm{C}(\sigma f)\|_{L^2(\overline{\mathbb{D}};\tau)} \leq \mathscr{C}\|f\|_{L^2(\mathbb{T};\sigma)},$$

if and only if these constants are finite:

$$\sigma(\mathbb{T})\cdot\tau(\overline{\mathbb{D}}) + \sup_{z\in\mathbb{D}}\{\mathrm{P}(\sigma\mathbf{1}_{\mathbb{T}\setminus I})(z)\mathrm{P}\tau(z) + \mathrm{P}\sigma(z)\mathrm{P}(\tau\mathbf{1}_{\overline{\mathbb{D}}\setminus B_I})(z)\} \equiv \mathscr{A}_2,$$

$$\sup_I \sigma(I)^{-1}\int_{B_I}|\mathrm{C}_\sigma\mathbf{1}_I(z)|^2\tau(dA(z)) \equiv \mathscr{T}^2,$$

$$\sup_I \tau(B_I)^{-1}\int_I|\mathrm{C}_\tau^*\mathbf{1}_{B_I}(w)|^2\sigma(dw) \equiv \mathscr{T}^2.$$

Finally, we have $\mathscr{C} \simeq \mathscr{A}_2^{1/2} + \mathscr{T}$.

Characterization of Carleson Measures for K_ϑ

Theorem (Lacey, Sawyer, Shen, Uriarte-Tuero, W.)

Let μ be a non-negative Borel measure supported on $\overline{\mathbb{D}}$ and let ϑ be an inner function on $\mathbb{D}$ with Clark measure σ. Set $\nu_{\mu,\vartheta} = |1-\vartheta|^2\mu$. The following are equivalent:

(i) *μ is a Carleson measure for K_ϑ, namely,*

$$\int_{\overline{\mathbb{D}}} |f(z)|^2 \, d\mu(z) \leq C(\mu)^2 \, \|f\|_{K_\vartheta}^2 \quad \forall f \in K_\vartheta;$$

(ii) *The Cauchy transform C is a bounded map between $L^2(\mathbb{T};\sigma)$ and $L^2(\overline{\mathbb{D}};\nu_{\mu,\vartheta})$, i.e., $\mathsf{C} : L^2(\mathbb{T};\sigma) \to L^2(\overline{\mathbb{D}};\nu_{\vartheta,\mu})$ is bounded;*

(iii) *The three conditions in the above theorem hold for the pair of measures σ and $\nu_{\mu,\vartheta}$. Moreover,*

$$C(\mu) \simeq \|\mathsf{C}\|_{L^2(\mathbb{T};\sigma)\to L^2(\overline{\mathbb{D}};\nu_{\vartheta,\mu})} \simeq \mathscr{A}_2^{1/2} + \mathscr{T}.$$

Danger!

Connection to Two-Weight Hilbert Transform

Recast the problem as a 'real-variable' question:

$$\mathrm{R}\sigma(x) \equiv \int_{\mathbb{R}} \frac{x-t}{|x-t|^2}\sigma(dt), \qquad x \in \mathbb{R}^2_+.$$

Write the coordinates of this operator as $(\mathrm{R}^1, \mathrm{R}^2)$. The second coordinate R^2 is the Poisson transform P. The Cauchy transform is

$$\mathrm{C}\sigma \equiv \mathrm{R}^1\sigma + i\mathrm{R}^2\sigma.$$

Question

Let σ denote a weight on $\mathbb{R}$ and τ denote a measure on the upper half plane $\mathbb{R}^2_+$. Find necessary and sufficient conditions on the pair of measures σ and τ so that the estimate below holds:

$$\|R_\sigma(f)\|_{L^2(\mathbb{R}^2_+;\tau)} = \|\mathrm{R}(\sigma f)\|_{L^2(\mathbb{R}^2_+;\tau)} \leq \mathscr{N}\,\|f\|_{L^2(\mathbb{R};\sigma)}.$$

Two Weight for Cauchy/Riesz Transforms

Theorem (Lacey, Sawyer, Shen, Uriarte-Tuero, W.)

Let σ be a weight on $\mathbb{R}$ and τ a weight on the closed upper half-plane $\mathbb{R}^2_+$. Then $\|R_\sigma(f)\|_{L^2(\mathbb{R}^2_+;\tau)} \le \mathscr{N}\,\|f\|_{L^2(\mathbb{R};\sigma)}$ if and only if for a finite positive constant $\mathscr{A}_2$ and $\mathscr{T}$,

$$\frac{\tau(Q_I)}{|I|} \times \int_{\mathbb{R}\setminus I} \frac{|I|}{(|I| + \mathrm{dist}(t, I))^2}\, \sigma(dt) \le \mathscr{A}_2,$$

$$\frac{\sigma(I)}{|I|} \times \int_{\mathbb{R}^2_+\setminus Q_I} \frac{|I|}{(|I| + \mathrm{dist}(x, Q_I))^2}\, \tau(dx) \le \mathscr{A}_2,$$

$$\int_{Q_I} |\mathrm{R}_\sigma \mathbf{1}_I(x)|^2\, \tau(dx) \le \mathscr{T}^2 \sigma(I) \quad \text{and} \quad \int_I |\mathrm{R}^*_\tau \mathbf{1}_{Q_I}(t)|^2\, \sigma(dt) \le \mathscr{T}^2 \tau(Q_I).$$

Moreover, $\mathscr{N} \simeq \mathscr{A}_2^{1/2} + \mathscr{T}$.

Observations about the Problem

- The kernel of this operator is one-dimensional:

$$\frac{x-t}{|x-t|^2}$$

Proofs and hypotheses should reflect this structure in some way.

- The necessity of the conditions is well-known:

$$\sup_I \frac{\tau(Q_I)}{|I|}\int_{\mathbb{R}\setminus I}\frac{|I|}{(|I|+\operatorname{dist}(t,I))^2}\,\sigma(dt)\le \mathscr{A}_2$$

$$\sup_I \frac{\sigma(I)}{|I|}\int_{\mathbb{R}^2_+\setminus Q_I}\frac{|I|}{(|I|+\operatorname{dist}(x,Q_I))^2}\,\tau(dx)\le \mathscr{A}_2$$

$$\sup_I \frac{1}{\sigma(I)}\int_{Q_I}|\mathsf{R}_\sigma \mathbf{1}_I(x)|^2\,\tau(dx)\le \mathscr{T}^2,$$

$$\sup_I \frac{1}{\tau(Q_I)}\int_I|\mathsf{R}_\tau^* \mathbf{1}_{Q_I}(t)|^2\,\sigma(dt)\le \mathscr{T}^2.$$

- Majority of efforts go into showing sufficiency of these conditions.

Necessary Conditions: Testing on Intervals/Cubes

Assuming that the Riesz transforms are bounded we have:

$$\|R_\sigma(f)\|_{L^2(\mathbb{R}^2_+;\tau)} = \|\mathrm{R}\,(\sigma f)\|_{L^2(\mathbb{R}^2_+;\tau)} \leq \mathscr{N}\,\|f\|_{L^2(\mathbb{R};\sigma)}\,.$$

A simple duality argument to show that:

$$\|R^*_\tau(f)\|_{L^2(\mathbb{R};\sigma)} = \|\mathrm{R}\,(\tau f)\|_{L^2(\mathbb{R};\sigma)} \leq \mathscr{N}\,\|f\|_{L^2(\mathbb{R}^2_+;\tau)}\,.$$

This implies:

$$\begin{aligned}
\int_{Q_I}|\mathrm{R}_\sigma \mathbf{1}_I(x)|^2\,\tau(dx) &\leq \|R_\sigma(1_I)\|^2_{L^2(\mathbb{R}^2_+;\tau)} \leq \mathscr{N}^2\,\|\mathbf{1}_I\|^2_{L^2(\mathbb{R};\sigma)} = \mathscr{N}^2\sigma(I).\\
\int_I|\mathrm{R}^*_\tau \mathbf{1}_{Q_I}(t)|^2\,\sigma(dt) &\leq \|R^*_\tau(1_{Q_I})\|^2_{L^2(\mathbb{R};\sigma)} \leq \mathscr{N}^2\,\|\mathbf{1}_{Q_I}\|^2_{L^2(\mathbb{R}^2_+;\tau)} = \mathscr{N}^2\tau(Q_I).
\end{aligned}$$

Which gives that $\mathscr{T} \leq \mathscr{N}$.

Necessary Conditions: Two Weight A_2

This is also a well-known argument. Both directions are similar and resort to testing on a function like:

$$p_I(x)^2 = \frac{|I|}{(|I| + \mathrm{dist}(x, I))^2}$$

Standard computations and estimates let one deduce:

$$\begin{aligned} \frac{\tau(Q_I)}{|I|}\left(\int_{\mathbb{R}\setminus I} \frac{|I|}{(|I| + \mathrm{dist}(t, I))^2}\,\sigma(dt)\right)^2 &\leq \|\mathrm{R}(\sigma p_I)\|^2_{L^2(\mathbb{R}^2_+;\tau)} \\ &\lesssim \mathscr{N}^2 \|p_I\|^2_{L^2(\mathbb{R}\setminus I;\sigma)}\,. \end{aligned}$$

Computations of this type prove that $\mathscr{A}_2^{\frac{1}{2}} \lesssim \mathcal{N}$. Which gives that $\mathscr{T} + \mathscr{A}_2^{\frac{1}{2}} \lesssim \mathcal{N}$.

Main Ideas behind the Proof

- Use "hidden" positivity to deduce the 'Energy Inequality':

$$
\begin{aligned}
\langle \mathsf{R}_\tau^* \varphi, h_J^\sigma \rangle_\sigma &= \iint_{\mathbb{R}^2_+ \setminus Q_J} \int_J \varphi(x) h_J^\sigma(t) \frac{x-t}{|x-t|^2}\, \sigma(dt)\, \tau(dx) \\
&= \iint_{\mathbb{R}^2_+ \setminus Q_J} \int_J \varphi(x) h_J^\sigma(t) \left(\frac{x-t}{|x-t|^2} - \frac{x-t_J}{|x-t_J|^2} \right) \sigma(dt)\, \tau(dx) \\
&\simeq \int_{\mathbb{R}^2_+} \frac{\varphi(y)\cdot y_2}{y_2^2 + (y_1 - (x_{Q_J})_1)^2 + (x_{Q_J})_2^2}\, \tau(dy) \cdot \left\langle \frac{t}{|J|}, h_{J'}^\sigma \right\rangle_\sigma \\
&\simeq \mathsf{T}_\tau \varphi(x_{Q_J}) \cdot \left\langle \frac{t}{|J|}, h_{J'}^\sigma \right\rangle_\sigma .
\end{aligned}
$$

- Replaces certain "off-diagonal" terms with "positive" operators.

Main Ideas behind the Proof

Lemma

For all intervals I_0 and partitions $\mathcal{I}$ of I_0 into dyadic intervals,

$$\sum_{I\in\mathcal{I}}\sum_{K\in\mathcal{W}I} \mathsf{T}_\tau(Q_{I_0}\setminus Q_K)(x_{Q_K})^2\left(\frac{1}{\sigma(I)}\sum_{\substack{J\,:\,J\subset I\\ J\text{ is good}}}\left\langle \frac{t}{|I|}, h_J^\sigma\right\rangle_\sigma^2\right)\sigma(K)\lesssim \mathscr{R}^2\tau(Q_{I_0}).$$

Lemma

The operator T_τ satisfies this two weight inequality:

$$\sum_{I\in\mathcal{I}}\sum_{K\in\mathcal{W}I} \mathsf{T}_\tau(\phi\cdot Q_I^c)(K)^2\mu_K\lesssim \mathscr{R}^2\|\phi\|^2_{L^2(\mathbb{R}^2_+;\tau)}.$$

Carleson Measures and Composition Operators

Let $\varphi : \mathbb{D} \to \mathbb{D}$ be holomorphic. The composition operator with symbol φ is $\mathfrak{C}_\varphi f = f \circ \varphi$.

Let τ be a weight on $\overline{\mathbb{D}}$, and define a Hilbert space of analytic functions by taking the closure of $H^\infty(\mathbb{D})$ with respect to the norm for $L^2(\overline{\mathbb{D}}; \tau)$. Call the resulting space H^2_τ.

To the function φ and weight τ we associate the pullback measure τ_φ defined as a measure on $\overline{\mathbb{D}}$, as $\tau_\varphi(E) \equiv \tau(\varphi^{-1}(E))$. Then

$$\|\mathfrak{C}_\varphi f\|^2_{H^2_\tau} = \int_{\overline{\mathbb{D}}} |f \circ \varphi(z)|^2 \, \tau(dA(z)) = \int_{\overline{\mathbb{D}}} |f(z)|^2 \, \tau_\varphi(dA(z)).$$

Behavior of the composition operator $\mathfrak{C}_\varphi : K_\vartheta \to H^2_\tau(\mathbb{D})$ is equivalent to

corresponding behavior of τ_φ as a Carleson measure for K_ϑ.

Bounded, Compact, Essential Norm of Composition Operators

Theorem (Lacey, Sawyer, Shen, Uriarte-Tuero, W.)

Let ϑ be an inner function. Let $\varphi : \mathbb{D} \to \mathbb{D}$ be analytic and let $\tau_\varphi = \tau \circ \varphi^{-1}$ denote the pullback measure associated to φ. The following are equivalent:

(i) $\mathfrak{C}_\varphi : K_\vartheta \to H^2_\tau$ *is bounded;*

(ii) τ_φ *is a Carleson measure for* K_ϑ, *namely,*

$$\int_{\overline{\mathbb{D}}} |f(z)|^2 \, \tau_\varphi(dA(z)) \leq C(\tau_\varphi)^2 \, \|f\|^2_{K_\vartheta} \quad \forall f \in K_\vartheta;$$

(iii) *The testing and* A_2 *conditions hold for the pair of weights* σ *on* $\mathbb{T}$ *and* $\nu_{\tau_\varphi,\vartheta} = |1-\vartheta|^2 \tau_\varphi$ *on* $\mathbb{D}$.

Compactness and essential norm can also be obtained from this result.

Safe Passage to the End!

Return to Beginning Conclusion Details

Commonalities of the Proof

- In both situations we are left studying the boundedness of an operator $T : L^2(u) \to L^2(v)$ (with the possibility that $u = v$).
- Proceed by duality to analyze the bilinear form: $\langle Tf, g\rangle_{L^2(v)}$.
- Without loss we can take the functions f and g supported on a large cube Q^0.
- Construct two independent dyadic lattices $\mathcal{D}_1$ and $\mathcal{D}_2$, one associated to f and the other to g.
    - In the case of the unit ball, the geometry dictates the grids.
    - In the model space case, the grid on $\mathbb{R}$ influences the construction of the grid in the upper half plane.
- Define expectation operators Δ_Q (Haar function on Q) and Λ (average on Q^0), then we have for every $f \in L^2(u)$

$$
\begin{aligned}
f &= \Lambda f + \sum_{Q\in\mathcal{D}_1} \Delta_Q f \\
\|f\|^2_{L^2(u)} &= \|\Lambda f\|^2_{L^2(u)} + \sum_{Q\in\mathcal{D}_1} \|\Delta_Q f\|^2_{L^2(u)} .
\end{aligned}
$$

Good and Bad Decomposition

- Define good and bad cubes. Heuristically, a cube $Q \in \mathcal{D}_1$ is *bad* if there is a cube $R \in \mathcal{D}_2$ of bigger size and Q is close to the boundary of R. More precisely, fix $0 < \delta < 1$ and $r \in \mathbb{N}$. $Q \in \mathcal{D}_1$ is said to be (δ, r)-*bad* if there is $R \in \mathcal{D}_2$ such that $|R| > 2^r|Q|$ and $\text{dist}(Q, \partial R) < |Q|^\delta |R|^{1-\delta}$.
- Decomposition of f and g into good and bad parts:

$$f = f_{good} + f_{bad}\,,\ \text{where } f_{good} = \Lambda f + \sum_{Q \in \mathcal{D}_1 \cap \mathcal{G}_1} \Delta_Q f$$

$$g = g_{good} + g_{bad}\,,\ \text{where } g_{good} = \Lambda g + \sum_{R \in \mathcal{D}_2 \cap \mathcal{G}_2} \Delta_R g.$$

- The probability that a cube is bad is small: $\mathbb{P}\{Q \text{ is bad}\} \leq \delta^2$ and $\mathbb{E}(\|f_{bad}\|_{L^2(u)}) \leq \delta \|f\|_{L^2(u)}$.
- Similar Statements for g hold as well.

Reduction to Controlling The Good Part

- Using the decomposition above, we have

$$\langle Tf, g\rangle_{L^2(v)} = \langle Tf_{good}, g_{good}\rangle_{L^2(v)} + R(f,g)$$

- Using the construction above, we have that

$$\mathbb{E}|R_\omega(f,g)| \leq 2\,\delta \|T\|_{L^2(u)\to L^2(v)} \|f\|_{L^2(u)} \|g\|_{L^2(v)}.$$

- Choosing δ small enough we only need to show that

$$\left|\langle Tf_{good}, g_{good}\rangle_{L^2(\mathbb{R}^d;\mu)}\right| \leq C\|f\|_{L^2(u)}\|g\|_{L^2(v)}.$$

- This will then give $\|T\|_{L^2(u)\to L^2(v)} \leq 2C$.

The fork in the road...

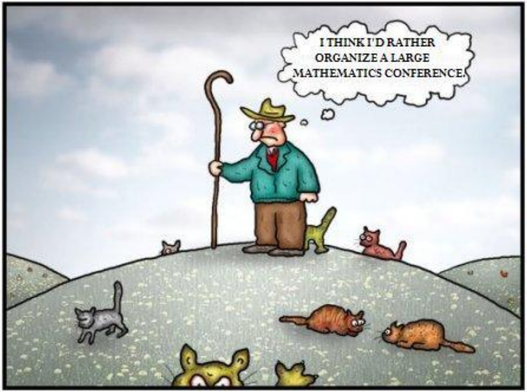
Estimating the Good Parts: The path diverges

- We then must control

$$\langle Tf_{good}, g_{good} \rangle_{L^2(v)}$$

- Reduce f_{good} and g_{good} to mean value zero by using the testing conditions.
- Control paraproduct type operators, use Carleson Embedding Theorem and the testing conditions to control terms.
- Control positive operators by the testing conditions and verifying the hypotheses for two-weight inequalities for positive operators.
- Proof strategies then diverge:
  - For the Cauchy transform mimic the the proof for the Hilbert transform with suitable modifications.
  - For the Besov-Sobolev space follow more standard $T1$ proof strategies with modifications.

◂ Conclusion



The daydreams of cat herders

(Modified from the Original Dr. Fun Comic)

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Thank You!

Comments & Questions