The Corona Problem

Brett D. Wick

Georgia Institute of Technology
School of Mathematics

AMS Spring Southeastern Sectional Meeting
Georgia Southern University
March 13th, 2011

This talk is based on joint work with:

Şerban Costea

Ecole Polytechnique
Federale de Lausanne

Eric T. Sawyer

McMaster University

Sergei R. Treil

Brown University

Talk Outline

- Motivation and History of the Problem
- Reformulation as the Operator Corona Problem
  - The Corona Problem and Bounded Analytic Projections
- The Corona Problem for Multiplier Algebras
  - Besov-Sobolev Spaces and Multiplier Algebras
  - The Baby Corona Problem & The Corona Problem
  - Toeplitz Corona Theorem
- Further Results and Future Directions

Where Did the Name Come From?

The Beer Problem?

Where Did the Name Come From?

The Corona Problem for $H^\infty(\mathbb{D})$

The Banach algebra $H^\infty(\mathbb{D})$ is the collection of all analytic functions on the disc such that

$$\|f\|_{H^\infty(\mathbb{D})} := \sup_{z\in\mathbb{D}} |f(z)| < \infty.$$

Let $\varphi : H^\infty(\mathbb{D}) \to \mathbb{C}$ be a multiplicative linear functional. Namely,

$$\varphi(fg) = \varphi(f)\varphi(g) \quad \text{and} \quad \varphi(f+g) = \varphi(f) + \varphi(g).$$

It's an easy exercise to show that for any multiplicative linear functional

$$\sup_{f\in H^\infty(\mathbb{D})} |\varphi(f)| \leq \|f\|_{H^\infty(\mathbb{D})}.$$

To each $z \in \mathbb{D}$ we can associate a multiplicative linear functional on $H^\infty(\mathbb{D})$:

$$\varphi_z(f) := f(z) \quad \text{(point evaluation at } z\text{)}.$$

Where Did the Name Come From?

The Corona Problem for $H^\infty(\mathbb{D})$

Every multiplicative linear functional φ determines a maximal (proper) ideal of $H^\infty(\mathbb{D})$: $\ker\varphi = \{f \in H^\infty(\mathbb{D}) : \varphi(f) = 0\}$.

Conversely, if M is a maximal (proper) ideal of $H^\infty(\mathbb{D})$ then $M = \ker\varphi$ for some multiplicative linear functional.

The maximal ideal space of $H^\infty(\mathbb{D})$, $\mathcal{M}_{H^\infty(\mathbb{D})}$, is the collection of all multiplicative linear functionals φ.

We then have that the maximal ideal space is contained in the unit ball of the dual space $H^\infty(\mathbb{D})$. If we put the weak-$*$ topology on this space then $\mathcal{M}_{H^\infty(\mathbb{D})}$ is a compact Hausdorff space.

The proceeding discussion then shows that $\mathbb{D} \subset \mathcal{M}_{H^\infty(\mathbb{D})}$.

Where Did the Name Come From?

The Corona Problem for $H^\infty(\mathbb{D})$

One then defines the Corona of $H^\infty(\mathbb{D})$ to be $\mathcal{M}_{H^\infty(\mathbb{D})} \setminus \overline{\mathbb{D}}$.

In 1941, Kakutani asked if there was a Corona in the maximal ideal space $\mathcal{M}_{H^\infty(\mathbb{D})}$ of $H^\infty(\mathbb{D})$, i.e. whether or not the disc $\mathbb{D}$ was dense in $\mathcal{M}_{H^\infty(\mathbb{D})}$?

Where Did the Name Come From?

The Corona Problem for $H^\infty(\mathbb{D})$

Using basic functional analysis, Kakutani's question can be phrased as the following question about analytic functions on the unit disc:

The open disc $\mathbb{D}$ is dense in $\mathcal{M}_{H^\infty(\mathbb{D})}$ (namely the algebra $H^\infty(\mathbb{D})$ has *no* Corona) if and only if the following condition holds:

If $f_1, \ldots, f_N \in H^\infty(\mathbb{D})$ and if

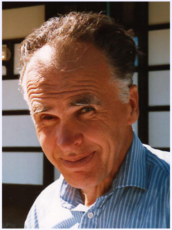
$$\max_{1 \leq j \leq N} |f_j(z)| \geq \delta > 0 \quad \forall z \in \mathbb{D}$$

then there exists $g_1, \ldots, g_N \in H^\infty(\mathbb{D})$ such that

$$1 = \sum_{j=1}^{N} f_j(z) g_j(z).$$

Where Did the Name Come From?

The Corona Problem for $H^\infty(\mathbb{D})$



Lennart Carleson

Kakutani's question was settled in 1962 by Carleson: $\overline{\mathbb{D}} = \mathcal{M}_{H^\infty(\mathbb{D})}$.

Theorem (Carleson's Corona Theorem)

Let $\{f_j\}_{j=1}^N \in H^\infty(\mathbb{D})$ satisfy

$$0 < \delta \leq \sum_{j=1}^N |f_j(z)|^2 \leq 1, \quad \forall z \in \mathbb{D}.$$

Then there are functions $\{g_j\}_{j=1}^N$ in $H^\infty(\mathbb{D})$ with

$$\sum_{j=1}^N f_j(z)\, g_j(z) = 1 \quad \forall z \in \mathbb{D} \text{ and } \|g_j\|_\infty \leq C_{\delta,N}.$$

Reasons to Care about the Corona Problem

1. Angles between Invariant Subspaces: $K_\theta := H^2(\mathbb{D}) \ominus \theta H^2(\mathbb{D})$
   The angle between K_{θ_1} and K_{θ_2} is positive if and only if there exists ψ_1 and ψ_2 such that
   $$\psi_1\theta_1 + \psi_2\theta_2 = 1.$$

2. Interpolation in $H^\infty(\mathbb{D})$: Let $B(z)$ be a Blaschke product with zeros $\{z_k\} = \mathcal{Z}$, and let $f \in H^\infty(\mathbb{D})$ be such that
   $$|f(z)| + |B(z)| \geq \delta > 0 \quad \forall z \in \mathbb{D}.$$
   Then there exists $p, q \in H^\infty(\mathbb{D})$ such that $pf + qB = 1$ if and only if there is a solution to the interpolation problem
   $$p(z_k) = \frac{1}{f(z_k)} \quad \forall z_k \in \mathcal{Z}.$$

3. Control Theory

Reformulation of the Problem

Let Ω be a domain in $\mathbb{C}^n$.
Let E and E_* be separable complex Hilbert spaces.
$H^\infty_{E_*\to E}(\Omega)$ is the collection of all bounded operator-valued functions.

$$F(z): E_* \to E \text{ and } \|F\|_{H^\infty_{E_*\to E}(\Omega)} := \sup_{z\in\Omega} \|F(z)\|_{E_*\to E}$$

Question ($H^\infty_{E_*\to E}(\Omega)$-Operator Corona Problem)

Let $F \in H^\infty_{E_*\to E}(\Omega)$. Can we find, preferably local, necessary and sufficient conditions on F so that it has an analytic left inverse? Namely, what conditions imply the existence of a function $G \in H^\infty_{E\to E_*}(\Omega)$ such that

$$G(z)F(z) \equiv I \quad \forall z \in \Omega.$$

A simple necessary condition is: $I \geq F^*(z)F(z) \geq \delta^2 I \quad \forall z \in \Omega$.

Connection to the Usual Corona Problem

Let $\Omega = \mathbb{D}$, the unit disc in the complex plane. Take $F(z) = (f_1(z), \ldots, f_N(z))^T$ in the Operator Corona Problem to recover:

Question (Corona Problem)

Suppose that $f_1, \ldots, f_N \in H^\infty(\mathbb{D})$ *with*

$$1 \geq \sum_{j=1}^{N} |f_j(z)|^2 \geq \delta > 0 \quad \forall z \in \mathbb{D}.$$

Do there exist $g_j \in H^\infty(\mathbb{D})$ *such that*

$$\sum_{j=1}^{N} f_j(z) g_j(z) \equiv 1 \quad \forall z \in \mathbb{D}?$$

Known Results for the disc $\mathbb{D}$

- When $E_* = \mathbb{C}$ and $\dim E < \infty$:
  - In 1962 Carleson demonstrated that the simple necessary condition is sufficient;
  - In 1967 Hörmander gave another proof of the result using $\overline{\partial}$-equations;
  - In 1979 Wolff gave a simpler proof of Carleson's result.
- When $E_* = \mathbb{C}, \dim E = \infty$:
  - Rosenblum, Tolokonnikov, and Uchiyama independently gave proofs.
- When $\dim E_* < \infty$ and $\dim E = \infty$: (Matrix Corona Problem)
  - Fuhrmann and Vasyunin independently demonstrated this.
- When $\dim E = \dim E_* = \infty$: (Operator Corona Problem)
  - In 1988 Treil constructed a counter example which indicates that the necessary condition is no longer sufficient.
  - In 2004 he gave another construction which demonstrated the same phenomenon.

Known Results for General Domains and Several Variables

- In 1970 Gamelin showed that the $H^\infty(\Omega)$-Corona problem is true when the domain $\Omega \subset \mathbb{C}$ has "holes."
- More generally, if $\Omega \subset \mathbb{C}$ is a Denjoy domain, then the Corona Theorem is true by a result of Garnett and Jones from 1985.

The Story is Much Different in Several Complex Variables

- Simple cases where the maximal ideal space of the algebra can be identified with a compact subset of $\mathbb{C}^n$. For example, the Ball Algebra $A(\mathbb{B}_n)$ or Polydisc Algebra $A(\mathbb{D}^n)$.
- There are counterexamples to Corona Theorems in several complex variables due to Cole, Sibony and Fornaess, but for domains that are very complicated geometrically.
- When $n \geq 2$ and $\Omega \subset \mathbb{C}^n$ (e.g. $\mathbb{B}_n$ or $\mathbb{D}^n$) the $H^\infty(\Omega)$-Corona Problem is open.

Take Away Point

*There are **NO** known domains in $\Omega \subset \mathbb{C}$ for which the $H^\infty(\Omega)$-Corona problem fails.*

*There are no **KNOWN** domains in $\Omega \subset \mathbb{C}^n$ for which the $H^\infty(\Omega)$-Corona problem holds.*

Nikolski's Lemma

The Operator Corona Problem is connected to complex geometry since the family of subspaces $\operatorname{Ran} F(z) = \{F(z)e : e \in E\}$ encode the information about this problem in a geometric manner.

Lemma (Nikolski's Lemma)

Let $F \in H^\infty_{E_* \to E}(\Omega)$ satisfy

$$I \geq F^*(z)F(z) \geq \delta^2 I, \quad \forall z \in \Omega.$$

Then F is left invertible in $H^\infty_{E_* \to E}(\Omega)$ (i.e., there exists $G \in H^\infty_{E \to E_*}(\Omega)$ such that $GF \equiv I$) if and only if there exists a function $\mathcal{P} \in H^\infty_{E \to E}(\Omega)$ whose values are projections (not necessarily orthogonal) onto $F(z)E$ for all $z \in \Omega$.

Key Point: Finding a left inverse is replaced with constructing a bounded analytic projection-valued function that takes a prescribed range.

The Corona Condition and Projections in $\mathbb{D}$

Let $F \in H^{\infty}_{E_* \to E}(\mathbb{D})$ be such that

$$I \geq F^*(z)F(z) \geq \delta^2 I \quad \forall z \in \mathbb{D}.$$

Set

$$\Pi(z) := F(z)(F^*(z)F(z))^{-1}F^*(z) \quad \forall z \in \mathbb{D}.$$

Note that

$$\begin{aligned} \Pi(z) &= \Pi(z)^* \\ \Pi^2(z) &= \Pi(z) \end{aligned}$$

The values of Π are orthogonal projections with $\operatorname{Ran}\Pi(z) = \operatorname{Ran}F(z)$ but clearly are **not analytic**. Direct computation demonstrates that

$$\Pi(z)\partial\Pi(z) = 0 \quad \forall z \in \mathbb{D}.$$

Solution to the Operator Corona Problem

Theorem (S. Treil, BDW, J. Amer. Math. Soc., 22 (2009))

Let $F \in H^\infty_{E_* \to E}(\mathbb{D})$ satisfy the Corona Condition $F^*F \geq \delta^2 I$. Assume that there exists a bounded non-negative subharmonic function φ such that

$$\Delta\varphi(z) \geq \|F'(z)\|^2 \quad z \in \mathbb{D}.$$

Then F has a holomorphic left inverse $G \in H^\infty_{E \to E_*}(\mathbb{D})$. Moreover, if the function φ satisfies

$$0 \leq \varphi(z) \leq K \quad \forall z \in \mathbb{D},$$

then one can find the left inverse G satisfying

$$\|G\|_\infty \leq \delta^{-1}\left(1 + 2\sqrt{(Ke^{K+1}+1)Ke^{K+1}}\right).$$

The Corona Problem and Bounded Analytic Projections

There is a more general theorem that can be obtained in this context:

Theorem (S. Treil, BDW, J. Amer. Math. Soc., 22 (2009))

Let $\Pi : \mathbb{D} \to B(E)$ be a $\mathcal{C}^2$ function whose values are orthogonal projections in E satisfying $\Pi \partial \Pi = 0$. Assume that there exists a bounded non-negative subharmonic function φ such that

$$\Delta \varphi(z) \geq \|\partial \Pi(z)\|^2 \qquad \forall z \in \mathbb{D}.$$

Then there exists a bounded analytic projection onto $\operatorname{Ran} \Pi(z)$, i.e., a function $\mathcal{P} \in H^\infty_{E \to E}$ such that $\mathcal{P}(z)$ is a projection onto $\operatorname{Ran} \Pi(z)$ for all $z \in \mathbb{D}$.

Moreover, if $0 \leq \varphi(z) \leq K$ for all $z \in \mathbb{D}$, then one can find $\mathcal{P}$ satisfying

$$\|\mathcal{P}\|_\infty \leq 1 + 2\sqrt{(Ke^{K+1} + 1)Ke^{K+1}}.$$

The Corona Condition and Projections

Method of Proof (of both Theorems):

- Need to construct a bounded analytic projection $\mathcal{P}(z)$ with

$$\operatorname{Ran}\mathcal{P}(z) = \operatorname{Ran}F(z) \quad \forall z \in \mathbb{D}.$$

- Use the projection $\Pi(z)$ as an initial guess for $\mathcal{P}(z)$.
- **Main Difficulty:** Find some bounded operator-valued function $V(z) : E \to E$ that we can use to "correct" the initial guess of $\Pi(z)$ to be holomorphic. Set $\mathcal{P}(z) = \Pi(z) - \Pi(z)V(z)(I - \Pi(z))$.
- The are several possible candidates for such a function:
  - Direct computation shows that $\varphi(z) = C\,\mathrm{tr}(F^*(z)F(z))$ works. Doesn't give good estimates in terms of the constants.
  - The function $\varphi(z) = \ln\det(F^*(z)F(z))$ also works and gives better estimates.
- The Corona Problem then can be solved by using Nikolski's Lemma to see that F is left invertible.

Extensions of the Corona Problem

The point of departure for many generalizations of Carleson's Corona Theorem is the following:

Observation

$H^\infty(\mathbb{D})$ is the (pointwise) multiplier algebra of the classical Hardy space $H^2(\mathbb{D})$ on the unit disc.

Namely, let $M_{H^2}(\mathbb{D})$ denote the class of functions φ such that

$$\|\varphi f\|_{H^2(\mathbb{D})} \leq C\|f\|_{H^2(\mathbb{D})}, \quad \forall f \in H^2(\mathbb{D}). \tag{$\dagger$}$$

with $\|\varphi\|_{M_{H^2}(\mathbb{D})} = \inf\{C : (\dagger) \text{ holds}\}$. Then $\varphi \in H^\infty(\mathbb{D})$ if and only if $\varphi \in M_{H^2}(\mathbb{D})$ and,

$$\|\varphi\|_{M_{H^2}(\mathbb{D})} = \|\varphi\|_{H^\infty(\mathbb{D})}.$$

Besov-Sobolev Spaces

The space $B_2^\sigma(\mathbb{B}_n)$ is the collection of holomorphic functions f on the unit ball $\mathbb{B}_n$ such that

$$\left\{ \sum_{k=0}^{m-1} \left| f^{(k)}(0) \right|^2 + \int_{\mathbb{B}_n} \left| \left(1 - |z|^2\right)^{m+\sigma} f^{(m)}(z) \right|^2 d\lambda_n(z) \right\}^{\frac{1}{2}} < \infty,$$

where $d\lambda_n(z) = \left(1 - |z|^2\right)^{-n-1} dV(z)$ is the invariant measure on $\mathbb{B}_n$ and $m + \sigma > \frac{n}{2}$.These spaces can also be defined for $1 < p < \infty$ with appropriate modifications.

Various choices of σ give important examples of classical function spaces:

- $\sigma = 0$: Corresponds to the Dirichlet Space;
- $\sigma = \frac{1}{2}$: Drury-Arveson Hardy Space;
- $\sigma = \frac{n}{2}$: Classical Hardy Space;
- $\sigma > \frac{n}{2}$: Bergman Spaces.

Besov-Sobolev Spaces

The spaces $B_2^\sigma(\mathbb{B}_n)$ are examples of reproducing kernel Hilbert spaces. Namely, for each point $\lambda \in \mathbb{B}_n$ there exists a function $k_\lambda \in B_2^\sigma(\mathbb{B}_n)$ such that

$$f(\lambda) = \langle f, k_\lambda \rangle_{B_2^\sigma}.$$

Easy computation to show that the kernel function k_λ is given by:

$$k_\lambda(z) = \frac{1}{\left(1 - \overline{\lambda} z\right)^{2\sigma}}$$

- $\sigma = \frac{1}{2}$: Drury-Arveson Hardy Space; $k_\lambda(z) = \frac{1}{1-\overline{\lambda}z}$
- $\sigma = \frac{n}{2}$: Classical Hardy Space; $k_\lambda(z) = \frac{1}{(1-\overline{\lambda}z)^{n}}$
- $\sigma = \frac{n+1}{2}$: Bergman Space; $k_\lambda(z) = \frac{1}{(1-\overline{\lambda}z)^{n+1}}$

Multiplier Algebras of Besov-Sobolev Spaces $M_{B_2^\sigma}(\mathbb{B}_n)$

We are interested in the multiplier algebras, $M_{B_2^\sigma}(\mathbb{B}_n)$, for $B_2^\sigma(\mathbb{B}_n)$. A function φ belongs to $M_{B_2^\sigma}(\mathbb{B}_n)$ if

$$\|\varphi f\|_{B_2^\sigma(\mathbb{B}_n)} \leq C\|f\|_{B_2^\sigma(\mathbb{B}_n)} \quad \forall f \in B_2^\sigma(\mathbb{B}_n)$$
$$\|\varphi\|_{M_{B_2^\sigma}(\mathbb{B}_n)} = \inf\{C : \text{above inequality holds}\}.$$

Let $\mathcal{X}_2^\sigma(\mathbb{B}_n)$ be the functions φ such that for all $f \in B_2^\sigma(\mathbb{B}_n)$:

$$\int_{\mathbb{B}_n} |f(z)|^2 \left|\left(1-|z|^2\right)^{m+\sigma} \varphi^{(m)}(z)\right|^2 d\lambda_n(z) \leq C\|f\|^2_{B_2^\sigma(\mathbb{B}_n)}, \qquad (\ddagger)$$

with $\|\varphi\|_{\mathcal{X}_2^\sigma(\mathbb{B}_n)} = \inf\{C : (\ddagger) \text{ holds}\}$. It is easy to see:

$$\begin{aligned} M_{B_2^\sigma}(\mathbb{B}_n) &= H^\infty(\mathbb{B}_n) \cap \mathcal{X}_2^\sigma(\mathbb{B}_n) \\ \|\varphi\|_{M_{B_2^\sigma}(\mathbb{B}_n)} &\approx \|\varphi\|_{H^\infty(\mathbb{B}_n)} + \|\varphi\|_{\mathcal{X}_2^\sigma(\mathbb{B}_n)}. \end{aligned}$$

The Corona Problem for $M_{B_2^\sigma}(\mathbb{B}_n)$

Question (Corona Problem for Multiplier Algebras $M_{B_2^\sigma}(\mathbb{B}_n)$)

Given $f_1, \ldots, f_N \in M_{B_2^\sigma}(\mathbb{B}_n)$ *satisfying*

$$0 < \delta \leq \sum_{j=1}^{N} |f_j(z)|^2 \leq 1 \quad \forall z \in \mathbb{B}_n.$$

Are there functions $g_1, \ldots, g_N \in M_{B_2^\sigma}(\mathbb{B}_n)$ *and a constant* $C_{n,\sigma,N,\delta}$ *such that:*

$$\begin{aligned} \sum_{j=1}^{N} \|g_j\|_{M_{B_2^\sigma}(\mathbb{B}_n)} &\leq C_{n,\sigma,N,\delta} \\ \sum_{j=1}^{N} g_j(z) f_j(z) &= 1 \quad \forall z \in \mathbb{B}_n? \end{aligned}$$

The Baby Corona Problem for $B_2^\sigma(\mathbb{B}_n)$

Question (Baby Corona Problem for $B_2^\sigma(\mathbb{B}_n)$)

Given $f_1, \ldots, f_N \in M_{B_2^\sigma}(\mathbb{B}_n)$ *satisfying*

$$0 < \delta \leq \sum_{j=1}^{N} |f_j(z)|^2 \leq 1 \quad \forall z \in \mathbb{B}_n$$

and $h \in B_2^\sigma(\mathbb{B}_n)$. *Does there exist a constant* $C_{n,\sigma,N,\delta}$ *and functions* $k_1, \ldots, k_N \in B_2^\sigma(\mathbb{B}_n)$ *satisfying*

$$\begin{aligned} \sum_{j=1}^{N} \|k_j\|_{B_2^\sigma(\mathbb{B}_n)}^2 &\leq C_{n,\sigma,N,\delta} \|h\|_{B_2^\sigma(\mathbb{B}_n)}^2, \\ \sum_{j=1}^{N} k_j(z) f_j(z) &= h(z) \quad \forall z \in \mathbb{B}_n? \end{aligned}$$

Corona implies Baby Corona

Indeed, if the Corona Problem is true and we take $h \in B_2^\sigma(\mathbb{B}_n)$, we can see that the Baby Corona Problem follows. Suppose $f_1, \ldots, f_N \in M_{B_2^\sigma}(\mathbb{B}_n)$, and there exists $g_1, \ldots, g_N \in M_{B_2^\sigma}(\mathbb{B}_n)$ such that

$$\sum_{j=1}^{N} \|g_j\|_{M_{B_2^\sigma}(\mathbb{B}_n)} \leq C_{n,\sigma,N,\delta} \qquad \sum_{j=1}^{N} g_j(z) f_j(z) = 1 \quad \forall z \in \mathbb{B}_n.$$

Multiplying the second equation by h, we find

$$h(z) = \sum_{j=1}^{N} g_j(z) f_j(z) h(z) = \sum_{j=1}^{N} k_j(z) f_j(z) \quad \forall z \in \mathbb{B}_n.$$

Since $g_1, \ldots, g_N \in M_{B_2^\sigma}(\mathbb{B}_n)$ we then have that $k_j := g_j h \in B_2^\sigma(\mathbb{B}_n)$ with $\|k_j\|_{B_2^\sigma(\mathbb{B}_n)} \leq \|g_j\|_{M_{B_2^\sigma}(\mathbb{B}_n)} \|h\|_{B_2^\sigma(\mathbb{B}_n)}$, so the claimed estimates follow as well.

Baby Corona implies Corona?

Toeplitz Corona Theorem

Theorem (Toeplitz Corona Theorem, (Agler and McCarthy))

Let $\mathcal{H}$ be a Hilbert function space in an open set Ω in $\mathbb{C}^n$ with an irreducible complete Nevanlinna-Pick kernel. Let $\epsilon > 0$ and let $f_1, \ldots, f_N \in M_{\mathcal{H}}$. Then the following are equivalent:

(i) *There exists $g_1, \ldots, g_N \in M_{\mathcal{H}}$ such that $\sum_{j=1}^N f_j g_j = 1$ and $\sum_{j=1}^N \|g_j\|_{M_{\mathcal{H}}} \leq \frac{1}{\epsilon}$;*

(ii) *For any $h \in \mathcal{H}$, there exists $k_1, \ldots, k_N \in \mathcal{H}$ such that $h = \sum_{j=1}^N k_j f_j$ and $\sum_{j=1}^N \|k_j\|_X^2 \leq \frac{1}{\epsilon^2}\|h\|_{\mathcal{H}}^2$.*

Moral: If the Hilbert space has a reproducing kernel with enough structure, then the Corona Problem and the Baby Corona Problem are the same question.

Baby Corona Theorem for $B_p^\sigma(\mathbb{B}_n)$

Theorem (Ş. Costea, E. Sawyer, BDW (Analysis & PDE to appear))

Let $0 \leq \sigma$ *and* $1 < p < \infty$. *Given* $f_1, \ldots, f_N \in M_{B_p^\sigma}(\mathbb{B}_n)$ *satisfying*

$$0 < \delta \leq \sum_{j=1}^{N} |f_j(z)|^2 \leq 1, \qquad z \in \mathbb{B}_n,$$

and $h \in B_p^\sigma(\mathbb{B}_n)$. *There are functions* $k_1, \ldots, k_N \in B_p^\sigma(\mathbb{B}_n)$ *and a constant* $C_{n,\sigma,N,p,\delta}$ *such that*

$$\begin{aligned} \sum_{j=1}^{N} \|k_j\|_{B_p^\sigma(\mathbb{B}_n)}^p &\leq C_{n,\sigma,N,p,\delta} \|h\|_{B_p^\sigma(\mathbb{B}_n)}^p, \\ \sum_{j=1}^{N} k_j(z) f_j(z) &= h(z) \quad \forall z \in \mathbb{B}_n. \end{aligned}$$

The Corona Theorem for $M_{B_2^\sigma}(\mathbb{B}_n)$

Theorem (Ş. Costea, E. Sawyer, BDW (Analysis & PDE to appear))

Let $0 \leq \sigma \leq \frac{1}{2}$. *Given* $f_1, \ldots, f_N \in M_{B_2^\sigma}(\mathbb{B}_n)$ *satisfying*

$$0 < \delta \leq \sum_{j=1}^{N} |f_j(z)|^2 \leq 1 \quad \forall z \in \mathbb{B}_n,$$

there are functions $g_1, \ldots, g_N \in M_{B_2^\sigma}(\mathbb{B}_n)$ *and a constant* $C_{n,\sigma,N,\delta}$ *such that*

$$\begin{aligned} \sum_{j=1}^{N} \|g_j\|_{M_{B_2^\sigma}(\mathbb{B}_n)} &\leq C_{n,\sigma,N,\delta} \\ \sum_{j=1}^{N} g_j(z) f_j(z) &= 1, \qquad z \in \mathbb{B}_n. \end{aligned}$$

The Corona Theorem for $M_{B_2^\sigma}(\mathbb{B}_n)$

The proof of this Theorem follows from the first Theorem very easily.

- When $0 \leq \sigma \leq \frac{1}{2}$ the spaces $B_2^\sigma(\mathbb{B}_n)$ are reproducing kernel Hilbert spaces with a complete Nevanlinna-Pick kernel.
- By the Toeplitz Corona Theorem, we then have that the Baby Corona Problem is equivalent to the full Corona Problem. The result then follows.

An additional corollary of the above result is the following:

Corollary

For $0 \leq \sigma \leq \frac{1}{2}$, the unit ball $\mathbb{B}_n$ is dense in the maximal ideal space of $M_{B_2^\sigma}(\mathbb{B}_n)$.

This is because the density of the unit ball $\mathbb{B}_n$ in the maximal ideal space of $M_{B_2^\sigma}(\mathbb{B}_n)$ is equivalent to the Corona Theorem above.

Sketch of Proof of the Baby Corona Theorem

Given $f_1, \dots, f_N \in M_{B_p^\sigma}(\mathbb{B}_n)$ satisfying

$$0 < \delta \le \sum_{j=1}^{N} |f_j(z)|^2 \le 1, \qquad z \in \mathbb{B}_n.$$

- Set $\varphi_j(z) = \frac{\overline{f_j(z)}}{\sum_j |f_j(z)|^2} h(z)$. We have that $\sum_{j=1}^N f_j(z)\varphi_j(z) = h(z)$.
- In order to have an analytic solution we will need to solve a sequence of $\overline{\partial}$-equations: The Koszul Complex.

  This gives an algorithmic way of solving the $\overline{\partial}$-equations for each $(0,q)$ with $1 \le q \le n$ after starting with a $(0,n)$ form.

  The Koszul Complex gives us $k_j = \varphi_j - \xi_j$.

  Algebraic properties of the Koszul complex give that $\sum_{j=1}^N f_j k_j = h$.

Sketch of Proof of the Baby Corona Theorem

Hard work then lets you conclude that the solutions obtained by the Koszul complex have the desired estimates.

Key Ideas in the Proof:

- Exact structure of the kernel of the solution operator that takes $(0, q)$ forms to $(0, q-1)$ forms:

$$\frac{(1 - w\overline{z})^{n-q}\left(1 - |w|^2\right)^{q-1}}{\triangle(w,z)^n}(\overline{w}_j - \overline{z}_j) \quad \forall\, 1 \leq q \leq n.$$

  Here $\triangle(w,z) = |1 - w\overline{z}|^2 - \left(1 - |w|^2\right)\left(1 - |z|^2\right)$.

- The solution operators to the $\overline{\partial}$-problem take the Besov-Sobolev spaces $B^p_\sigma(\mathbb{B}_n)$ to themselves.

The $H^\infty(\mathbb{B}_n)$ Corona Problem

When $\sigma = \frac{n}{2}$ (the classical Hardy space $H^2(\mathbb{B}_n)$), we have a weaker version of the Corona Problem that we can prove.

Theorem (Ş. Costea, E. Sawyer, BDW (J. Funct. Anal., 258 (2010)))

Given $f_1, \ldots, f_N \in H^\infty(\mathbb{B}_n)$ satisfying $0 < \delta \leq \sum_{j=1}^N |f_j(z)|^2 \leq 1$ for all $z \in \mathbb{B}_n$. Then there is a constant $C_{n,N,\delta}$ and there are functions $g_1, \ldots, g_N \in BMOA(\mathbb{B}_n)$ satisfying

$$\begin{aligned} \sum_{j=1}^N \|g_j\|_{BMOA(\mathbb{B}_n)} &\leq C_{n,N,\delta} \\ \sum_{j=1}^N g_j(z) f_j(z) &= 1, \qquad z \in \mathbb{B}_n. \end{aligned}$$

This gives another proof of a famous theorem of Varopoulos.

Open Problems and Future Directions

1. Does the algebra $H^\infty(\mathbb{B}_n)$ of bounded analytic functions on the ball have a Corona in its maximal ideal space?
   Prize for Solution: A Case of Corona Beer

2. Can we prove a Corona Theorem for **any** algebra in higher dimensions that is not the multiplier algebra of a Hilbert space with the complete Nevanlinna-Pick property? Any $\frac{1}{2} < \sigma \leq \frac{n}{2}$ would be extremely interesting.

Thank You!