Corona Theorems for Multiplier Algebras on the Unit Ball

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Where Did the Name Come From?

The Corona Problem for $H^\infty(\mathbb{D})$

The Banach algebra $H^\infty(\mathbb{D})$ is the collection of all analytic functions on the disc such that

$$\|f\|_{H^\infty(\mathbb{D})} \equiv \sup_{z\in\mathbb{D}} |f(z)| < \infty.$$

Let $\varphi : H^\infty(\mathbb{D}) \to \mathbb{C}$ be a non-trivial multiplicative linear functional. Namely,

$$\varphi(fg) = \varphi(f)\varphi(g) \quad \text{and} \quad \varphi(f+g) = \varphi(f) + \varphi(g).$$

It's an easy exercise to show that for any multiplicative linear functional

$$\sup_{f\in H^\infty(\mathbb{D})} |\varphi(f)| \leq \|f\|_{H^\infty(\mathbb{D})}.$$

To each $z \in \mathbb{D}$ we can associate a multiplicative linear functional on $H^\infty(\mathbb{D})$:

$$\varphi_z(f) \equiv f(z) \quad \text{(point evaluation at } z\text{)}.$$

Where Did the Name Come From?

The Corona Problem for $H^\infty(\mathbb{D})$

Every non-trivial multiplicative linear functional φ determines a maximal (proper) ideal of $H^\infty(\mathbb{D})$: $\ker\varphi = \{f \in H^\infty(\mathbb{D}) : \varphi(f) = 0\}$.

Conversely, if M is a maximal (proper) ideal of $H^\infty(\mathbb{D})$ then $M = \ker\varphi$ for some non-trivial multiplicative linear functional.

The maximal ideal space of $H^\infty(\mathbb{D})$, $\mathcal{M}_{H^\infty(\mathbb{D})}$, can then be identified with the collection of all non-trivial multiplicative linear functionals φ.

We then have that the maximal ideal space is contained in the unit ball of the dual space $H^\infty(\mathbb{D})$. If we put the weak-$*$ topology on this space then $\mathcal{M}_{H^\infty(\mathbb{D})}$ is a compact Hausdorff space.

The proceeding discussion then shows that $\mathbb{D} \subset \mathcal{M}_{H^\infty(\mathbb{D})}$.

Where Did the Name Come From?

The Corona Problem for $H^\infty(\mathbb{D})$

One then defines the Corona of $H^\infty(\mathbb{D})$ to be $\mathcal{M}_{H^\infty(\mathbb{D})} \setminus \overline{\mathbb{D}}^{w-*}$.

In 1941, Kakutani asked if there was a Corona in the maximal ideal space $\mathcal{M}_{H^\infty(\mathbb{D})}$ of $H^\infty(\mathbb{D})$, i.e. whether or not the disc $\mathbb{D}$ was dense in $\mathcal{M}_{H^\infty(\mathbb{D})}$?

Where Did the Name Come From?

The Corona Problem for $H^\infty(\mathbb{D})$

Using basic functional analysis, Kakutani's question can be phrased as the following question about analytic functions on the unit disc:

The open disc $\mathbb{D}$ is dense in $\mathcal{M}_{H^\infty(\mathbb{D})}$ (namely the algebra $H^\infty(\mathbb{D})$ has *no* Corona) if and only if the following condition holds:

If $f_1, \ldots, f_N \in H^\infty(\mathbb{D})$ and if

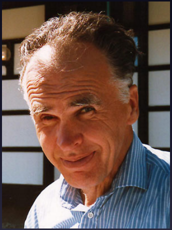
$$\max_{1 \leq j \leq N} |f_j(z)| \geq \delta > 0 \quad \forall z \in \mathbb{D}$$

then there exists $g_1, \ldots, g_N \in H^\infty(\mathbb{D})$ such that

$$1 = \sum_{j=1}^{N} f_j(z) g_j(z).$$

Where Did the Name Come From?

The Corona Problem for $H^\infty(\mathbb{D})$



Lennart Carleson

Kakutani's question was settled in 1962 by Lennart Carleson: $\overline{\mathbb{D}}^{w-*} = \mathcal{M}_{H^\infty(\mathbb{D})}$.

Theorem (Carleson's Corona Theorem)

Let $\{f_j\}_{j=1}^N \in H^\infty(\mathbb{D})$ *satisfy*

$$0 < \delta \leq \sum_{j=1}^N |f_j(z)|^2 \leq 1 \quad \forall z \in \mathbb{D}.$$

Then there are functions $\{g_j\}_{j=1}^N$ *in* $H^\infty(\mathbb{D})$ *with*

$$\sum_{j=1}^N f_j(z)\, g_j(z) = 1 \; \forall z \in \mathbb{D} \text{ and } \|g_j\|_{H^\infty(\mathbb{D})} \leq C_{\delta,N}.$$

Known Results for $H^\infty(\mathbb{D})$

- Finitely many generators:
  - In 1962 Carleson demonstrated that the simple necessary condition is sufficient;
  - In 1967 Hörmander gave another proof of the result using $\overline{\partial}$-equations;
  - In 1979 Wolff gave a simpler proof of Carleson's result.
- Infinitely many generators:
  - Rosenblum, Tolokonnikov, and Uchiyama independently gave proofs.
- The Matrix Corona Problem:
  - Fuhrmann and Vasyunin independently demonstrated this.
- Operator Corona Problem:
  - In 1988 Treil constructed a counterexample which indicates that the necessary condition is no longer sufficient.
  - In 2004 he gave another construction which demonstrated the same phenomenon.

Known Results for Several Complex Variables

- The Story is Much Different in Several Complex Variables.
- Simple cases where the maximal ideal space of the algebra can be identified with a compact subset of $\mathbb{C}^n$. For example, the Ball Algebra $A(\mathbb{B}_n)$ or Polydisc Algebra $A(\mathbb{D}^n)$.
- There are counterexamples to Corona Theorems in several complex variables due to Cole, Sibony and Fornaess, but for domains that are pseudoconvex except at one point.

Take Away Point

*There are no **KNOWN** domains in $X \subset \mathbb{C}^n$ for which the $H^\infty(X)$-Corona problem **holds**.*

Conjecture (Corona Problem for the Polydisc or Ball)

When $n \geq 2$ and X is either the $\mathbb{B}_n$ or $\mathbb{D}^n$ the $H^\infty(X)$-Corona Problem is true.

Baby Corona Problem for Hilbert Spaces

- $X \subset \mathbb{C}^n$ a domain;
- $\mathcal{H}$ a reproducing kernel Hilbert space of analytic functions on X;
- $M_{\mathcal{H}}$ the collection of multipliers of $\mathcal{H}$:

$$M_{\mathcal{H}} = \{\varphi \in \mathcal{H} : \varphi h \in \mathcal{H} \quad \forall h \in \mathcal{H}\}$$

Problem (Baby Corona Problem)

Given $f_1, \ldots, f_N \in M_{\mathcal{H}}$ satisfying

$$1 \geq |f_1(z)|^2 + \cdots + |f_N(z)|^2 \geq \delta > 0, \qquad z \in X,$$

and $h \in \mathcal{H}$. Are there functions $l_1, \ldots, l_N \in \mathcal{H}$ and a constant $C > 0$ such that

$$\begin{aligned} \|l_1\|_{\mathcal{H}}^2 + \cdots + \|l_N\|_{\mathcal{H}}^2 &\leq C\,\|h\|_{\mathcal{H}}^2, \\ l_1(z)\,f_1(z) + \cdots + l_N(z)\,f_N(z) &= h(z) \quad \forall z \in X? \end{aligned}$$

Corona Problem for Multiplier Algebras

Problem (Corona Problem)

Given $f_1, \ldots, f_N \in M_{\mathcal{H}}$ *satisfying*

$$1 \geq |f_1(z)|^2 + \cdots + |f_N(z)|^2 \geq \delta > 0 \quad \forall z \in X.$$

Are there functions $g_1, \ldots, g_N \in M_{\mathcal{H}}$ *and a constant* $C > 0$ *such that*

$$\begin{aligned} \|g_1\|_{M_{\mathcal{H}}}^2 + \cdots + \|g_N\|_{M_{\mathcal{H}}}^2 &\leq C, \\ g_1(z) f_1(z) + \cdots + g_N(z) f_N(z) &= 1 \quad \forall z \in X? \end{aligned}$$

Corona Problem always implies the Baby Corona Problem. Indeed, multiplying the second equation by h, we find

$$\sum_{j=1}^{N} h(z) g_j(z) f_j(z) = h(z) \quad \forall z \in X; \sum_{j=1}^{N} l_j(z) f_j(z) = h(z) \quad \forall z \in X.$$

Where $l_j \equiv g_j h$ and $\|l_j\|_{\mathcal{H}} \leq \|g_j\|_{M_{\mathcal{H}}} \|h\|_{\mathcal{H}}$.

Baby Corona implies Corona?

Complete Nevanlinna-Pick Kernels

Definition

A kernel k is a *complete Nevanlinna-Pick* (CNP) kernel for $\mathcal{H}$ if for any finite set of n distinct points $\{x_1, \ldots, x_n\} \subset X$ the matrix

$$\left(1 - \frac{k_{x_i}(x_n)k_{x_n}(x_j)}{k_{x_i}(x_j)k_{x_n}(x_n)}\right)_{i,j=1}^{n-1}$$

is positive semi-definite.

Example (Kernels on $\mathbb{D}$)

$k_z(w) = (1 - \overline{z}w)^{-1}$ *is CNP. But,* $k_z(w) = (1 - \overline{z}w)^{-2}$ *is not.*

Example (Kernels on $\mathbb{B}_n$)

$k_z(w) = (1 - \overline{z}w)^{-1}$ *is CNP. But,* $k_z(w) = (1 - \overline{z}w)^{-n}$ *is not.*

Baby Corona implies Corona?

Toeplitz Corona Theorem

Theorem (Toeplitz Corona Theorem, (Agler and McCarthy))

Let $\mathcal{H}$ be a Hilbert function space on an open set X in $\mathbb{C}^n$ with an irreducible complete Nevanlinna-Pick kernel. Let $\epsilon > 0$ and let $f_1, \ldots, f_N \in M_{\mathcal{H}}$. Then the following are equivalent:

- (i) *There exists $g_1, \ldots, g_N \in M_{\mathcal{H}}$ such that $\sum_{j=1}^N f_j g_j = 1$ and $\sum_{j=1}^N \|g_j\|_{M_{\mathcal{H}}} \leq \frac{1}{\epsilon}$;*
- (ii) *For any $h \in \mathcal{H}$, there exists $l_1, \ldots, l_N \in \mathcal{H}$ such that $h = \sum_{j=1}^N l_j f_j$ and $\sum_{j=1}^N \|l_j\|_{\mathcal{H}}^2 \leq \frac{1}{\epsilon^2} \|h\|_{\mathcal{H}}^2$.*

Moral: If the Hilbert space has a reproducing kernel with enough structure, then the Corona Problem and the Baby Corona Problem are the same question.

The $\overline{\partial}$-problem and the Koszul Complex

First consider the case of two functions so that we can see the connections between this problem and the $\overline{\partial}$-problem we will study. Suppose we have two functions $f_1, f_2 \in M_{\mathcal{H}}$ such that

$$1 \geq |f_1(z)|^2 + |f_2(z)|^2 \geq \delta \quad \forall z \in X.$$

Define the following functions

$$\varphi_1(z) \equiv \frac{\overline{f_1(z)}}{|f_1(z)|^2 + |f_2(z)|^2} \quad \varphi_2(z) \equiv \frac{\overline{f_2(z)}}{|f_1(z)|^2 + |f_2(z)|^2}.$$

The hypotheses on f_1 and f_2 imply that the functions φ_1 and φ_2 are in fact bounded and smooth on X. Note that

$$\varphi_1(z)f_1(z) + \varphi_2(z)f_2(z) = 1 \quad \forall z \in X$$

but the functions φ_1 and φ_2 are in general *not* analytic.

Motivating the $\overline{\partial}$-problem

Now, observe for any function r we have that the functions

$$g_1 = \varphi_1 + rf_2 \quad g_2 = \varphi_2 - rf_1$$

also solve the problem

$$f_1 g_1 + f_2 g_2 = 1.$$

Our goal is to select a good choice of function r so that the resulting choice will make g_1 and g_2 be analytic and belong to the multiplier algebra. Now, we have that g_1 is analytic if and only if

$$0 = \overline{\partial} g_1 = \overline{\partial} \varphi_1 + f_2 \overline{\partial} r.$$

Similarly, g_2 is analytic if and only if

$$0 = \overline{\partial} g_2 = \overline{\partial} \varphi_2 - f_1 \overline{\partial} r.$$

Using these two equations and the condition that $f_1\varphi_1 + f_2\varphi_2 = 1$ gives that the function r must satisfy the equation

$$\overline{\partial} r = \varphi_1 \overline{\partial} \varphi_2 - \varphi_2 \overline{\partial} \varphi_1.$$

The Koszul Complex

- If $f = (f_j)_{j=1}^N$ satisfies $|f|^2 = \sum_{j=1}^N |f_j|^2 \geq 1$, let
$$\Omega_0^1 = \frac{\overline{f}}{|f|^2} = \left(\frac{\overline{f_j}}{|f|^2}\right)_{j=1}^N = \left(\Omega_0^1(j)\right)_{j=1}^N,$$
which we view as a 1-tensor (in $\mathbb{C}^N$) of $(0,0)$-forms.

- Then $\varphi = \Omega_0^1 h$ satisfies $f \cdot g = h$, but in general fails to be analytic.

- The Koszul complex provides a scheme when f and h are holomorphic for solving a sequence of $\overline{\partial}$ equations that result in a correction term $\Lambda_f \Gamma_0^2$ that when subtracted from φ above yields an *analytic* solution to $f \cdot g = h$.

Lifting of Forms

- The 1-tensor of $(0,1)$-forms $\overline{\partial}\Omega_0 = \left(\overline{\partial}\frac{\overline{f_j}}{|f|^2}\right)_{j=1}^{N} = \left(\overline{\partial}\Omega_0^1(j)\right)_{j=1}^{N}$ is given by

$$\overline{\partial}\Omega_0^1(j) = \overline{\partial}\frac{\overline{f_j}}{|f|^2} = \frac{1}{|f|^4}\sum_{k=1}^{N} f_k \overline{\{f_k \partial f_j - \partial f_k f_j\}}.$$

- A key fact is that this 1-tensor of $(0,1)$-forms can be "factored" as

$$\overline{\partial}\Omega_0^1 = \Lambda_f \Omega_1^2 \equiv \left[\sum_{k=1}^{N} \Omega_1^2(j,k) f_k\right]_{j=1}^{N},$$

where the 2-tensor Ω_1^2 of $(0,1)$-forms is given by

$$\Omega_1^2 = \left[\Omega_1^2(j,k)\right]_{j,k=1}^{N} = \left[\frac{\overline{\{f_k \partial f_j - \partial f_k f_j\}}}{|f|^4}\right]_{j,k=1}^{N}.$$

Solving the complex ...

- We can repeat this process and by induction we have

$$\overline{\partial}\Omega_q^{q+1} = \Lambda_f \Omega_{q+1}^{q+2}, \qquad 0 \leq q \leq n,$$

where Ω_q^{q+1} is an *alternating* $(q+1)$-tensor of $(0,q)$-forms.

- Recall that h is holomorphic. When $q = n$ we have that $\Omega_n^{n+1} h$ is $\overline{\partial}$-closed since every $(0,n)$-form is $\overline{\partial}$-closed.

- This allows us to begin solving a chain of $\overline{\partial}$ equations

$$\overline{\partial}\Gamma_{q-2}^{q} = \Omega_{q-1}^{q} h - \Lambda_f \Gamma_{q-1}^{q+1}$$

...using that the forms are closed

- Since $\Omega_n^{n+1}h$ is $\overline{\partial}$-closed and alternating, there is an alternating $(n+1)$-tensor Γ_{n-1}^{n+1} of $(0,n-1)$-forms satisfying

$$\overline{\partial}\Gamma_{n-1}^{n+1} = \Omega_n^{n+1}h.$$

- Now note that the n-tensor $\Omega_{n-1}^{n}h - \Lambda_f\Gamma_{n-1}^{n+1}$ of $(0,n-1)$-forms is $\overline{\partial}$-closed:

$$\begin{aligned}\overline{\partial}\left(\Omega_{n-1}^{n}h - \Lambda_f\Gamma_{n-1}^{n+1}\right) &= \overline{\partial}\Omega_{n-1}^{n}h - \overline{\partial}\Lambda_f\Gamma_{n-1}^{n+1}\\ &= \Lambda_f\Omega_n^{n+1}h - \Lambda_f\Omega_n^{n+1}h = 0.\end{aligned}$$

- Thus there is an alternating n-tensor Γ_{n-2}^{n} of $(0,n-2)$-forms satisfying:

$$\overline{\partial}\Gamma_{n-2}^{n} = \Omega_{n-1}^{n}h - \Lambda_f\Gamma_{n-1}^{n+1}.$$

The Bezout Equation

- With the convention that $\Gamma_n^{n+2} \equiv 0$, induction shows that there are alternating $(q+2)$-tensors Γ_q^{q+2} of $(0,q)$-forms for $0 \leq q \leq n$ satisfying

$$\begin{aligned} \overline{\partial}\left(\Omega_q^{q+1} h - \Lambda_f \Gamma_q^{q+2}\right) &= 0, \qquad 0 \leq q \leq n, \\ \overline{\partial}\Gamma_{q-1}^{q+1} &= \Omega_q^{q+1} h - \Lambda_f \Gamma_q^{q+2}, \qquad 1 \leq q \leq n. \end{aligned}$$

- Now

$$g \equiv \Omega_0^1 h - \Lambda_f \Gamma_0^2$$

is holomorphic by the first line above, and since Γ_0^2 is antisymmetric, we compute that $\Lambda_f \Gamma_0^2 \cdot f = \Gamma_0^2(f,f) = 0$ and

$$f \cdot g = \Omega_0^1 h \cdot g - \Lambda_f \Gamma_0^2 \cdot f = h - 0 = h.$$

- Thus $g = (g_1, g_2, \ldots, g_N)$ is an N-vector of holomorphic functions satisfying $f \cdot g = h$.

Besov-Sobolev Spaces

The space $B_2^\sigma(\mathbb{B}_n)$ is the collection of holomorphic functions f on the unit ball $\mathbb{B}_n$ such that

$$\left\{ \sum_{k=0}^{m-1} \left| f^{(k)}(0) \right|^2 + \int_{\mathbb{B}_n} \left| \left(1 - |z|^2\right)^{m+\sigma} f^{(m)}(z) \right|^2 d\lambda_n(z) \right\}^{\frac{1}{2}} < \infty,$$

where $d\lambda_n(z) = \left(1 - |z|^2\right)^{-n-1} dV(z)$ is the invariant measure on $\mathbb{B}_n$ and $m + \sigma > \frac{n}{2}$. These spaces can also be defined for $1 < p < \infty$ with appropriate modifications.
Various choices of σ give important examples of classical function spaces:

- $\sigma = 0$: Corresponds to the Dirichlet Space;
- $\sigma = \frac{1}{2}$: Drury-Arveson Hardy Space;
- $\sigma = \frac{n}{2}$: Classical Hardy Space;
- $\sigma > \frac{n}{2}$: Bergman Spaces.

Besov-Sobolev Spaces

The spaces $B_2^\sigma(\mathbb{B}_n)$ are examples of reproducing kernel Hilbert spaces. Namely, for each point $\lambda \in \mathbb{B}_n$ there exists a function $k_\lambda \in B_2^\sigma(\mathbb{B}_n)$ such that

$$f(\lambda) = \langle f, k_\lambda \rangle_{B_2^\sigma}.$$

Easy computations show that the kernel function k_λ is given by:

$$k_\lambda(z) = \frac{1}{\left(1 - \overline{\lambda} z\right)^{2\sigma}}$$

- $\sigma = \frac{1}{2}$: Drury-Arveson Hardy Space; $k_\lambda(z) = \frac{1}{1-\overline{\lambda}z}$
- $\sigma = \frac{n}{2}$: Classical Hardy Space; $k_\lambda(z) = \frac{1}{(1-\overline{\lambda}z)^n}$
- $\sigma = \frac{n+1}{2}$: Bergman Space; $k_\lambda(z) = \frac{1}{(1-\overline{\lambda}z)^{n+1}}$

When $0 \leq \sigma \leq \frac{1}{2}$ then k_λ is a complete Nevanlinna-Pick kernel.

Baby Corona Theorem for $B_p^\sigma(\mathbb{B}_n)$

Theorem (Ş. Costea, E. Sawyer, BDW; Analysis & PDE **4** (2011))

Let $0 \leq \sigma$ *and* $1 < p < \infty$. *Given* $f_1, \ldots, f_N \in M_{B_p^\sigma}(\mathbb{B}_n)$ *satisfying*

$$0 < \delta \leq \sum_{j=1}^{N} |f_j(z)|^2 \leq 1, \qquad z \in \mathbb{B}_n,$$

and $h \in B_p^\sigma(\mathbb{B}_n)$. *There are functions* $k_1, \ldots, k_N \in B_p^\sigma(\mathbb{B}_n)$ *and a constant* $C_{n,\sigma,N,p,\delta}$ *such that*

$$\begin{aligned} \sum_{j=1}^{N} \|k_j\|_{B_p^\sigma(\mathbb{B}_n)}^p &\leq C_{n,\sigma,N,p,\delta} \|h\|_{B_p^\sigma(\mathbb{B}_n)}^p, \\ \sum_{j=1}^{N} k_j(z) f_j(z) &= h(z) \quad \forall z \in \mathbb{B}_n. \end{aligned}$$

The Corona Theorem for $M_{B_2^\sigma}(\mathbb{B}_n)$

Theorem (Ş. Costea, E. Sawyer, BDW; Analysis & PDE 4 (2011))

Let $0 \leq \sigma \leq \frac{1}{2}$ *and* $p = 2$. *Given* $f_1, \ldots, f_N \in M_{B_2^\sigma}(\mathbb{B}_n)$ *satisfying*

$$0 < \delta \leq \sum_{j=1}^{N} |f_j(z)|^2 \leq 1 \quad \forall z \in \mathbb{B}_n.$$

There are functions $g_1, \ldots, g_N \in M_{B_2^\sigma}(\mathbb{B}_n)$ *and a constant* $C_{n,\sigma,N,\delta}$ *such that*

$$\begin{aligned} \sum_{j=1}^{N} \|g_j\|_{M_{B_2^\sigma}(\mathbb{B}_n)} &\leq C_{n,\sigma,N,\delta} \\ \sum_{j=1}^{N} g_j(z) f_j(z) &= 1, \qquad z \in \mathbb{B}_n. \end{aligned}$$

Easy Corollaries

Corollary

For $0 \leq \sigma \leq \frac{1}{2}$, the unit ball $\mathbb{B}_n$ is dense in the maximal ideal space of $M_{B_2^\sigma}(\mathbb{B}_n)$.

Theorem (Ş. Costea, E. Sawyer, BDW)

For $1 < p < \infty$ and $0 \leq \sigma < \infty$ the Taylor spectrum for the tuple $f \in \left(B_p^\sigma(\mathbb{B}_n)\right)^m$ is given by $\sigma\left(f, B_\sigma^p(\mathbb{B}_n)\right) = \overline{f(\mathbb{B}_n)}$.

Sketch of Proof of the Baby Corona Theorem

Given $f_1, \ldots, f_N \in M_{B_p^\sigma}(\mathbb{B}_n)$ satisfying

$$0 < \delta \leq \sum_{j=1}^{N} |f_j(z)|^2 \leq 1, \qquad z \in \mathbb{B}_n.$$

- Set $\varphi_j(z) = \frac{\overline{f_j(z)}}{\sum_j |f_j(z)|^2} h(z)$. We have that $\sum_{j=1}^N f_j(z)\varphi_j(z) = h(z)$.
- This solution is smooth and satisfies the correct estimates, but is far from analytic.
- In order to have an analytic solution we utilize the Koszul complex to produce analytic solutions g such that $\sum_{j=1}^N f_j(z) g_j(z) = h(z)$.
- However, the estimates we seek are now unfortunately in doubt. We need to show that solutions to $\overline{\partial}$-problems "preserve" $B_p^\sigma(\mathbb{B}_n)$ norms.

Estimates in B_p^σ and $\overline{\partial}$-problems

Given a form $(0,q)$-form η we can solve $\overline{\partial} u = \eta$ by

$$u(z) = \int_{\mathbb{B}_n} C_{(0,q)}(\xi, z) \wedge \eta(\xi)$$

- The exact structure of the kernel of the solution operator that takes $(0,q)$ forms to $(0,q-1)$ forms is known. Key part of the kernel is:

$$\frac{(1 - w\overline{z})^{n-q}\left(1 - |w|^2\right)^{q-1}}{\triangle(w,z)^n}(\overline{w}_j - \overline{z}_j) \quad \forall\, 1 \leq q \leq n.$$

  Here $\triangle(w,z) = |1 - w\overline{z}|^2 - \left(1 - |w|^2\right)\left(1 - |z|^2\right)$.
- One then needs to show that these solution operators map the Besov-Sobolev spaces $B_\sigma^p(\mathbb{B}_n)$ to themselves. This is accomplished by a couple of key facts.

Key Properties in Proving the Estimates

- The Besov-Sobolev spaces are very “flexible” in terms of the norm that one can use. One need only take the parameter m sufficiently high.
- One can show that these solution operators are very well behaved on “real variable” versions of the space $B_\sigma^p(\mathbb{B}_n)$. These, of course, contain the space that we are interested in.
- To show that the solution operators are bounded on $L^p(\mathbb{B}_n; dV)$ the original proof uses the Schur Test. To handle the boundedness on $B_\sigma^p(\mathbb{B}_n)$, we can also use the Schur test but this requires a little more work to handle the derivative.
- The fact that the Corona data belongs to the multiplier algebra allows one to prove certain embedding theorems that are used to control terms in the application of the Schur test.
- Hard work (and technical estimates that we omit!) then lets you conclude that the solutions obtained by the Koszul complex have the desired estimates.

The $H^\infty(\mathbb{B}_n)$ Corona Problem

For the classical Hardy space $H^2(\mathbb{B}_n)$, (i.e., $\sigma = \frac{n}{2}$), we have:

Theorem (Ş. Costea, E. Sawyer, BDW; J. Funct. Anal. 258 (2010))

Given $f_1, \ldots, f_N \in H^\infty(\mathbb{B}_n)$ satisfying $0 < \delta \le \sum_{j=1}^N |f_j(z)|^2 \le 1$ for all $z \in \mathbb{B}_n$. Then there is a constant $C_{n,N,\delta}$ and there are functions $g_1, \ldots, g_N \in BMOA(\mathbb{B}_n)$ satisfying

$$\begin{aligned} \sum_{j=1}^N \|g_j\|_{BMOA(\mathbb{B}_n)} &\le C_{n,N,\delta} \\ \sum_{j=1}^N g_j(z) f_j(z) &= 1, \qquad z \in \mathbb{B}_n. \end{aligned}$$

This gives another proof of a famous theorem of Varopoulos.

Baby Corona for $H^\infty(\mathbb{B}_n)$ versus Corona for $H^\infty(\mathbb{B}_n)$

We know that the Corona Problem always implies the Baby Corona Problem. By the Toeplitz Corona Theorem, we know that, under certain conditions on the reproducing kernel, these problems are in fact equivalent. But, what happens if we don't have these conditions?

Theorem (Equivalence between Corona and Baby Corona, (Amar 2003))

Let $\{g_j\}_{j=1}^N \subseteq H^\infty(\mathbb{B}_n)$. Then there exists $\{f_j\}_{j=1}^N \subseteq H^\infty(\mathbb{B}_n)$ with

$$\sum_{j=1}^N f_j(z) g_j(z) = 1 \quad \forall z \in \mathbb{B}_n \quad \text{and} \quad \sum_{j=1}^N \|g_j\|_{H^\infty(\mathbb{B}_n)} \leq \frac{1}{\delta}$$

if and only if

$$\mathcal{M}_g^\mu (\mathcal{M}_g^\mu)^* \geq \delta^2 I_\mu$$

for all probability measures μ on $\partial\mathbb{B}_n$.

Baby Corona for $H^\infty(\mathbb{B}_n)$ versus Corona for $H^\infty(\mathbb{B}_n)$

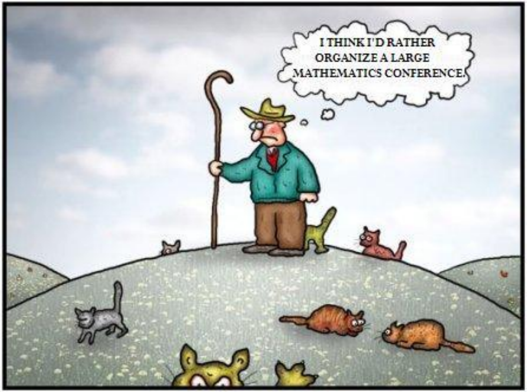
This is a great theorem since it suggests how to attack the Corona Problem for $H^\infty(\mathbb{B}_n)$. But, the difficulty is that one must solve the Baby Corona Problem for **every** probability measure on $\partial\mathbb{B}_n$. Instead, it is possible to reduce this to a class of probability measures for which the methods of harmonic analysis and operator theory are more amenable.

Theorem (Trent, BDW; Complex Anal.Oper. Theory 3 (2009))

Assume that $\mathcal{M}_g^H \mathcal{M}_g^{H*} \geq \delta^2 I_H$ *for all* $H \in \mathcal{W}$. *Then there exists a* $f_1, \ldots, f_N \in H^\infty(\mathbb{B}_n)$, *so that*

$$\sum_{j=1}^N f_j(z) g_j(z) = 1 \; \forall z \in \mathbb{B}_n \quad \text{and} \quad \sum_{j=1}^N \|f_j\|_{H^\infty(\mathbb{B}_n)} \leq \frac{1}{\delta}.$$

This reduces the $H^\infty(\mathbb{B}_n)$ Corona Problem to a certain "weighted" Baby Corona Problem.



The daydreams of cat herders

(Modified from the Original Dr. Fun Comic)

Thanks to Lyudmyla, Maria, and Sandra for the Invitation and Organizing the Meeting!

Thank You!

Tack Så Mycket!