Composition of Haar Paraproducts

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Sarason's Conjecture

- $H^2(\mathbb{D})$, the $L^2(\mathbb{T})$ closure of the analytic polynomials on $\mathbb{D}$.
- $\mathbb{P} : L^2(\mathbb{T}) \to H^2(\mathbb{D})$ be the orthogonal projection.
- A Toeplitz operator with symbol φ is the following map from $H^2(\mathbb{D}) \to H^2(\mathbb{D})$:

$$T_\varphi(f) \equiv \mathbb{P}\left(\varphi f\right).$$

- An important question raised by Sarason is the following:

Conjecture (Sarason Conjecture)

The composition of $T_\varphi T_{\overline{\psi}}$ is bounded on $H^2(\mathbb{D})$ if and only if

$$\sup_{z\in\mathbb{D}} \left(\int_{\mathbb{T}} \frac{1-|z|^2}{\left|1-z\overline{\xi}\right|^2} \left|\varphi(\xi)\right|^2 dm(\xi)\right) \left(\int_{\mathbb{T}} \frac{1-|z|^2}{\left|1-z\overline{\xi}\right|^2} \left|\psi(\xi)\right|^2 dm(\xi)\right) < \infty$$

Unfortunately, this is not true! A counterexample was constructed by Nazarov.

The Sarason Conjecture & Hilbert Transform

Question (Sarason Question (Revised Version))

Obtain necessary and sufficient (testable (?)) conditions so that one can tell if $T_\varphi T_{\overline{\psi}}$ is bounded on $H^2(\mathbb{D})$ by evaluating these conditions.

Possible to rephrase this question as one about the two-weight boundedness of the Hilbert transform.

- Let M_ϕ denote multiplication by ϕ: $M_\phi f \equiv \phi f$;
- $H^2(|\phi|^2)$ is the $L^2(\mathbb{T})$ closure of $p\phi$ where p is an analytic polynomial;

$$\begin{array}{ccc} H^2 & \stackrel{T_\varphi T_{\overline{\psi}}}{\longrightarrow} & H^2 \\ M_{\overline{\psi}} \downarrow & & \downarrow M_\varphi \\ L^2\left(\mathbb{T}; |\psi|^{-2}\right) & \stackrel{H}{\longrightarrow} & L^2\left(\mathbb{T}; |\varphi|^2\right) \end{array}$$

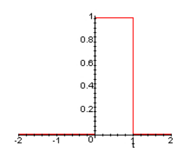
Deep work by Nazarov, Treil, Volberg, and then subsequent work by Lacey, Sawyer, Shen, Uriarte-Tuero allow for an answer in terms of the Hilbert transform.

Haar Paraproducts

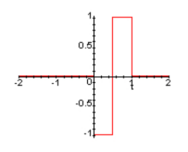
- $L^2 \equiv L^2(\mathbb{R})$;
- $\mathcal{D}$ is the standard grid of dyadic intervals on $\mathbb{R}$;
- Define the Haar function h_I^0 and averaging function h_I^1 by

$$h_I^0 \equiv h_I \equiv \frac{1}{\sqrt{|I|}}\left(-\mathbf{1}_{I_-} + \mathbf{1}_{I_+}\right) \quad I \in \mathcal{D}$$

$$h_I^1 \equiv \frac{1}{|I|}\mathbf{1}_I \quad I \in \mathcal{D}.$$



$h^1_{[0,1]}(x)$



$h^0_{[0,1]}(x)$

- $\{h_I\}_{I\in\mathcal{D}}$ is an orthonormal basis of L^2.

Haar Paraproducts from Multiplication Operators

Given a function b and f it is possible to study their pointwise product by expanding in their Haar series:

$$\begin{aligned} bf &= \left(\sum_{I\in\mathcal{D}} \langle b, h_I\rangle_{L^2} h_I\right)\left(\sum_{J\in\mathcal{D}} \langle f, h_J\rangle_{L^2} h_J\right) \\ &= \sum_{I,J\in\mathcal{D}} \langle b, h_I\rangle_{L^2} \langle f, h_J\rangle_{L^2} h_I h_J \\ &= \left(\sum_{I=J} + \sum_{I\subsetneq J} + \sum_{J\subsetneq I}\right) \langle b, h_I\rangle_{L^2} \langle f, h_J\rangle_{L^2} h_I h_J \\ &= \sum_{I\in\mathcal{D}} \langle b, h_I\rangle_{L^2} \langle f, h_I\rangle_{L^2} h_I^1 + \sum_{I\in\mathcal{D}} \langle b, h_I\rangle_{L^2} \left\langle f, h_I^1\right\rangle_{L^2} h_I \\ &\quad + \sum_{I\in\mathcal{D}} \left\langle b, h_I^1\right\rangle_{L^2} \langle f, h_I\rangle_{L^2} h_I. \end{aligned}$$

Haar Paraproducts

Definition (Haar Paraproducts)

Given a symbol sequence $b = \{b_I\}_{I\in\mathcal{D}}$ and a pair $(\alpha,\beta) \in \{0,1\}^2$, define the *dyadic paraproduct* acting on a function f by

$$\mathsf{P}_b^{(\alpha,\beta)} f \equiv \sum_{I\in\mathcal{D}} b_I \left\langle f, h_I^{\beta} \right\rangle_{L^2} h_I^{\alpha}.$$

The index (α,β) is referred to as the *type* of $\mathsf{P}_b^{(\alpha,\beta)}$.

Question (Discrete Sarason Question)

For each choice of pairs $(\alpha,\beta),(\epsilon,\delta) \in \{0,1\}^2$, obtain necessary and sufficient conditions on symbols b and d so that

$$\left\| \mathsf{P}_b^{(\alpha,\beta)} \circ \mathsf{P}_d^{(\epsilon,\delta)} \right\|_{L^2\to L^2} < \infty.$$

Internal Cancellations and Simple Characterizations

When there are internal zeros the behavior of $\mathsf{P}_b^{(\alpha,0)} \circ \mathsf{P}_d^{(0,\beta)}$reduces to the behavior of $P_a^{(\alpha,\beta)}$ for a special symbol a. For $f, g \in L^2$, let $f \otimes g : L^2 \to L^2$ be the map given by

$$f \otimes g(h) \equiv f \left\langle g, h \right\rangle_{L^2}.$$

Then:

$$\begin{aligned} \mathsf{P}_b^{(\alpha,0)} \circ \mathsf{P}_d^{(0,\beta)} &= \left(\sum_{I \in \mathcal{D}} b_I h_I^{\alpha} \otimes h_I \right) \left(\sum_{J \in \mathcal{D}} d_J h_J \otimes h_J^{\beta} \right) \\ &= \sum_{I \in \mathcal{D}} b_I d_I \, h_I^{\alpha} \otimes h_I^{\beta} \\ &= P_{b \circ d}^{(\alpha,\beta)}. \end{aligned}$$

Here $b \circ d$ is the Schur product of the symbols, i.e., $(b \circ d)_I = b_I d_I$.

Norms and Induced Sequences

For a sequence $a = \{a_I\}_{I\in\mathcal{D}}$ define the following quantities:

$$\begin{aligned} \|a\|_{\ell^\infty} &\equiv \sup_{I\in\mathcal{D}} |a_I| ; \\ \|a\|_{CM} &\equiv \sqrt{\sup_{I\in\mathcal{D}} \frac{1}{|I|} \sum_{J\subset I} |a_J|^2} . \end{aligned}$$

Associate to $\{a_I\}_{I\in\mathcal{D}}$ two additional sequences indexed by $\mathcal{D}$:

$$\begin{aligned} E(a) &\equiv \left\{ \frac{1}{|I|} \sum_{J\subset I} a_J \right\}_{I\in\mathcal{D}} ; \\ \widehat{S}(a) &\equiv \left\{ \left\langle \sum_{J\in\mathcal{D}} a_J h_J^1, h_I \right\rangle_{L^2} \right\}_{I\in\mathcal{D}} = \left\{ \sum_{J\subsetneq I} a_J \widehat{h_J^1}(I) \right\}_{I\in\mathcal{D}} . \end{aligned}$$

Classical Characterizations

Theorem (Characterizations of Type $(0,0)$, $(0,1)$, and $(1,0)$)

The following characterizations are true:

$$\begin{aligned}\left\|\mathsf{P}_a^{(0,0)}\right\|_{L^2\to L^2} &= \|a\|_{\ell^\infty} ;\\ \left\|\mathsf{P}_a^{(0,1)}\right\|_{L^2\to L^2} &= \left\|\mathsf{P}_a^{(1,0)}\right\|_{L^2\to L^2} \approx \|a\|_{CM} .\end{aligned}$$

$$\mathsf{P}_a^{(1,1)} = \mathsf{P}_{\widehat{S}(a)}^{(1,0)} + \mathsf{P}_{\widehat{S}(a)}^{(0,1)} + \mathsf{P}_{E(a)}^{(0,0)} .$$

Theorem (Characterization of Type $(1,1)$)

The operator norm $\left\|\mathsf{P}_a^{(1,1)}\right\|_{L^2\to L^2}$ *of* $\mathsf{P}_a^{(1,1)}$ *on* L^2 *satisfies*

$$\left\|\mathsf{P}_a^{(1,1)}\right\|_{L^2\to L^2} \approx \left\|\widehat{S}(a)\right\|_{CM} + \|E(a)\|_{\ell^\infty} .$$

Proof of the Easy Characterizations

A simple computation gives

$$\begin{aligned} \left\| \mathsf{P}_a^{(0,0)} f \right\|_{L^2}^2 &= \sum_{I,I'\in\mathcal{D}} a_I \overline{a_{I'}} \left\langle f, h_I \right\rangle_{L^2} \overline{\left\langle f, h_{I'} \right\rangle_{L^2}} \left\langle h_I, h_{I'} \right\rangle_{L^2} \\ &= \sum_{I\in\mathcal{D}} \left| a_I \right|^2 \left| \left\langle f, h_I \right\rangle_{L^2} \right|^2, \\ \left\| f \right\|_{L^2}^2 &= \sum_{I\in\mathcal{D}} \left| \left\langle f, h_I \right\rangle_{L^2} \right|^2. \end{aligned}$$

Similarly,

$$\begin{aligned} \left\| \mathsf{P}_a^{(0,1)} f \right\|_{L^2}^2 &= \sum_{I,I'\in\mathcal{D}} a_I \overline{a_{I'}} \left\langle f, h_I^1 \right\rangle_{L^2} \overline{\left\langle f, h_{I'}^1 \right\rangle_{L^2}} \left\langle h_I, h_{I'} \right\rangle_{L^2} \\ &= \sum_{I\in\mathcal{D}} \left| a_I \right|^2 \left| \left\langle f, h_I^1 \right\rangle_{L^2} \right|^2 \end{aligned}$$

Proof of the Easy Characterizations

Theorem (Carleson Embedding Theorem)

Let $\{\alpha_I\}_{I\in\mathcal{D}}$ be positive constants. The following two statements are equivalent:

$$\begin{aligned}\sum_{I\in\mathcal{D}} \alpha_I \left\langle f, h_I^1 \right\rangle_{L^2}^2 &\lesssim C\|f\|_{L^2}^2 \quad \forall f \in L^2;\\ \sup_{I\in\mathcal{D}} \frac{1}{|I|}\sum_{J\subset I} \alpha_J &\leq C.\end{aligned}$$

Then the Carleson Embedding Theorem gives

$$\left\|\mathsf{P}_a^{(0,1)}\right\|_{L^2\to L^2} \lesssim \|a\|_{CM}.$$

Let $\hat{I}$ denote the parent of I, and then we have

$$\left\|\mathsf{P}_a^{(0,1)}\right\|_{L^2\to L^2}^2 \geq \left\|\mathsf{P}_a^{(0,1)} h_{\hat{I}}\right\|_{L^2}^2 \gtrsim \frac{1}{|I|}\sum_{J\subset I} |a_J|^2.$$

Decomposing $(1,1)$ into Pieces

Expand the averaging functions in a Haar series:

$$h_I^1 = \sum_{J \supsetneq I} \left\langle h_I^1, h_J \right\rangle_{L^2} h_J,$$

$$\begin{aligned}
\mathsf{P}_a^{(1,1)} f &= \sum_{I \in \mathcal{D}} a_I \left\langle f, h_I^1 \right\rangle_{L^2} h_I^1 \\
&= \sum_{I \in \mathcal{D}} a_I \left\langle f, \left(\sum_{J \supsetneq I} \left\langle h_I^1, h_J \right\rangle_{L^2} h_J \right) \right\rangle_{L^2} \left(\sum_{K \supsetneq I} \left\langle h_I^1, h_K \right\rangle_{L^2} h_K \right) \\
&= \left\{ \sum_{J \subsetneq K} + \sum_{K \subsetneq J} + \sum_{J = K} \right\} \sum_{I \subset J \cap K} a_I \left\langle h_I^1, h_J \right\rangle_{L^2} \left\langle h_I^1, h_K \right\rangle_{L^2} \langle f, h_J \rangle_{L^2} h \\
&\equiv \mathsf{P}_{\widehat{S}(a)}^{(1,0)} f + \mathsf{P}_{\widehat{S}(a)}^{(0,1)} f + \mathsf{P}_{E(a)}^{(0,0)} f.
\end{aligned}$$

Alternate Interpretations: Testing Conditions

It is easy to see for paraproducts of type $(0,0)$ that:

$$\begin{aligned}\left\|\mathsf{P}_a^{(0,0)}\right\|_{L^2\to L^2} &= \|a\|_{\ell^\infty} \\ &= \sup_{I\in\mathcal{D}}\left\|\mathsf{P}_a^{(0,0)}h_I\right\|_{L^2}.\end{aligned}$$

Moreover,

$$\begin{aligned}\left\|\mathsf{P}_a^{(1,0)}\right\|_{L^2\to L^2} &= \left\|\mathsf{P}_a^{(0,1)}\right\|_{L^2\to L^2} \\ &\approx \|a\|_{CM} \\ &\approx \sup_{I\in\mathcal{D}}\left\|\mathsf{P}_a^{(0,1)}h_I\right\|_{L^2}.\end{aligned}$$

These observations suggest seeking a characterization for the other compositions in terms of testing conditions on classes of functions.

Two Weight Inequalities in Harmonic Analysis

Given weights u and v on $\mathbb{R}$ and an operator T a problem one frequently encounters in harmonic analysis is the following:

Question

Determine necessary and sufficient conditions on T, u, and v so that

$$T : L^2(\mathbb{R}; u) \to L^2(\mathbb{R}; v)$$

is bounded.

Meta-Theorem (Characterization of Boundedness via Testing)

The operator $T : L^2(\mathbb{R}; u) \to L^2(\mathbb{R}; u)$ *is bounded if and only if*

$$\begin{aligned} \|T(u\mathbf{1}_Q)\|_{L^2(v)} &\lesssim \|\mathbf{1}_Q\|_{L^2(u)} \\ \|T^*(v\mathbf{1}_Q)\|_{L^2(u)} &\lesssim \|\mathbf{1}_Q\|_{L^2(v)}. \end{aligned}$$

Characterization of Type $(0,1,1,0)$

For a sequence a, and interval $I \in \mathcal{D}$ let $\mathsf{Q}_I a \equiv \sum_{J \subset I} a_J h_J$.

Theorem (E. Sawyer, S. Pott, M. Reguera-Rodriguez, BDW)

The composition $\mathsf{P}_b^{(0,1)} \circ \mathsf{P}_d^{(1,0)}$ *is bounded on* L^2 *if and only if both*

$$\begin{aligned} \left\| \mathsf{Q}_I \mathsf{P}_b^{(0,1)} \mathsf{P}_d^{(1,0)} \left(\mathsf{Q}_I \overline{d} \right) \right\|_{L^2}^2 &\leq C_1^2 \left\| \mathsf{Q}_I d \right\|_{L^2}^2 \quad \forall I \in \mathcal{D}; \\ \left\| \mathsf{Q}_I \mathsf{P}_d^{(0,1)} \mathsf{P}_b^{(1,0)} \left(\mathsf{Q}_I \overline{b} \right) \right\|_{L^2}^2 &\leq C_2^2 \left\| \mathsf{Q}_I b \right\|_{L^2}^2 \quad \forall I \in \mathcal{D}. \end{aligned}$$

Moreover, the norm of $\mathsf{P}_b^{(0,1)} \circ \mathsf{P}_d^{(1,0)}$ *on* L^2 *satisfies*

$$\left\| \mathsf{P}_b^{(0,1)} \circ \mathsf{P}_d^{(1,0)} \right\|_{L^2 \to L^2} \approx C_1 + C_2$$

where C_1 *and* C_2 *are the best constants appearing above.*

Rephrasing the Testing Conditions

We want to rephrase the testing conditions on $\mathsf{Q}_I\overline{d}$ and $\mathsf{Q}_I\overline{b}$:

$$\begin{aligned}\left\|\mathsf{Q}_I\mathsf{P}_b^{(0,1)}\mathsf{P}_d^{(1,0)}\left(\mathsf{Q}_I\overline{d}\right)\right\|_{L^2}^2 &\leq C_1^2\left\|\mathsf{Q}_I d\right\|_{L^2}^2 \quad \forall I\in\mathcal{D};\\ \left\|\mathsf{Q}_I\mathsf{P}_d^{(0,1)}\mathsf{P}_b^{(1,0)}\left(\mathsf{Q}_I\overline{b}\right)\right\|_{L^2}^2 &\leq C_2^2\left\|\mathsf{Q}_I b\right\|_{L^2}^2 \quad \forall I\in\mathcal{D}.\end{aligned}$$

It isn't hard to see that these are equivalent to the following inequalities on the sequences:

$$\begin{aligned}\sum_{J\subset I}\left|b_J\right|^2\frac{1}{\left|J\right|^2}\left(\sum_{L\subset J}\left|d_L\right|^2\right)^2 &\leq C_1^2\sum_{L\subset I}\left|d_L\right|^2 \quad \forall I\in\mathcal{D};\\ \sum_{J\subset I}\left|d_J\right|^2\frac{1}{\left|J\right|^2}\left(\sum_{L\subset J}\left|b_L\right|^2\right)^2 &\leq C_2^2\sum_{L\subset I}\left|b_L\right|^2 \quad \forall I\in\mathcal{D}.\end{aligned}$$

Characterization of Type $(0, 1, 0, 0)$

Theorem (E. Sawyer, S. Pott, M. Reguera-Rodriguez, BDW)

The composition $\mathsf{P}_b^{(0,1)} \circ \mathsf{P}_d^{(0,0)}$ *is bounded on* L^2 *if and only if both*

$$\begin{aligned} |d_I|^2 \left\| \mathsf{P}_b^{(0,1)} h_I \right\|_{L^2}^2 &\leq C_1^2 \quad \forall I \in \mathcal{D}; \\ \left\| \mathsf{Q}_I \mathsf{P}_d^{(0,0)} \mathsf{P}_b^{(1,0)} \mathsf{Q}_I \overline{b} \right\|_{L^2}^2 &\leq C_2^2 \left\| \mathsf{Q}_I b \right\|_{L^2}^2 \quad \forall I \in \mathcal{D}. \end{aligned}$$

Moreover, the norm of $\mathsf{P}_b^{(0,1)} \circ \mathsf{P}_d^{(0,0)}$ *on* L^2 *satisfies*

$$\left\| \mathsf{P}_b^{(0,1)} \circ \mathsf{P}_d^{(0,0)} \right\|_{L^2 \to L^2} \approx C_1 + C_2$$

where C_1 *and* C_2 *are the best constants appearing above.*

Rephrasing Testing Conditions

Again, it is possible to recast the conditions:

$$\begin{aligned} |d_I|^2 \left\| \mathsf{P}_b^{(0,1)} h_I \right\|_{L^2}^2 &\leq C_1^2 \quad \forall I \in \mathcal{D}; \\ \left\| \mathsf{Q}_I \mathsf{P}_d^{(0,0)} \mathsf{P}_b^{(1,0)} \mathsf{Q}_I \overline{b} \right\|_{L^2}^2 &\leq C_2^2 \left\| \mathsf{Q}_I b \right\|_{L^2}^2 \quad \forall I \in \mathcal{D} \end{aligned}$$

as expressions depending only on the sequences. In particular, these are equivalent to the following inequalities:

$$\begin{aligned} \frac{|d_I|^2}{|I|} \sum_{L \subsetneq I} |b_L|^2 &\leq C_1^2 \quad \forall I \in \mathcal{D}; \\ \sum_{J \subset I} \frac{|d_J|^2}{|J|} \left(\sum_{K \subset J_+} |b_K|^2 - \sum_{K \subset J_-} |b_K|^2 \right)^2 &\leq C_2^2 \sum_{L \subset I} |b_L|^2 \quad \forall I \in \mathcal{D}. \end{aligned}$$

Preliminaries

For $I \in \mathcal{D}$ set

$$T(I) \equiv I \times \left[\frac{|I|}{2}, |I|\right] \qquad \text{(Carleson Tile);}$$

$$Q(I) \equiv I \times [0, |I|] = \bigcup_{J \subset I} T(J) \qquad \text{(Carleson Square).}$$

- The dyadic lattice $\mathcal{D}$ is in correspondence with the Carleson Tiles.
- Let $\mathcal{H}$ denote the upper half plane $\mathbb{C}_+$: $\mathcal{H} = \bigcup_{I \in \mathcal{D}} T(I)$.
- For a non-negative function σ let $L^2(\mathcal{H}; \sigma)$ denote the functions that are square integrable with respect to $\sigma\, dA$, i.e,

$$\|f\|^2_{L^2(\mathcal{H};\sigma)} \equiv \int_{\mathcal{H}} |f(z)|^2 \sigma(z)\, dA(z) < \infty.$$

  When $\sigma \equiv 1$, $L^2(\mathcal{H}; 1) \equiv L^2(\mathcal{H})$.
- For $f \in L^2(\mathcal{H})$, let $\widetilde{f} \equiv \frac{f}{\|f\|_{L^2(\mathcal{H})}}$ denote the normalized function.

Functions Constant on Tiles

Let $L^2_c(\mathcal{H}) \subset L^2(\mathcal{H})$ be the subspace of functions which are constant on tiles. Namely, $f : \mathcal{D} \to \mathbb{C}$

$$f = \sum_{I \in \mathcal{D}} f_I \mathbf{1}_{T(I)}.$$

Then

$$\begin{aligned} L^2_c(\mathcal{H}) &\equiv \left\{ f : \mathcal{D} \to \mathbb{C} : \sum_{I \in \mathcal{D}} |f(I)|^2 |I|^2 < \infty \right\}; \\ \|f\|^2_{L^2_c(\mathcal{H})} &\equiv \frac{1}{2} \sum_{I \in \mathcal{D}} |f(I)|^2 |I|^2 . \end{aligned}$$

Easy to show:

$\left\{ \widetilde{\mathbf{1}}_{T(I)} \right\}_{I \in \mathcal{D}}$ is an orthonormal basis of $L^2_c(\mathcal{H})$;

$\left\{ \widetilde{\mathbf{1}}_{Q(I)} \right\}_{I \in \mathcal{D}}$ is an Riesz basis of $L^2_c(\mathcal{H})$.

The Gram Matrix of $\mathsf{P}_b^{(0,1)} \circ \mathsf{P}_d^{(1,0)}$

Let $\mathfrak{G}_{\mathsf{P}_b^{(0,1)} \circ \mathsf{P}_d^{(1,0)}} = [G_{I,J}]_{I,J\in\mathcal{D}}$ be the Gram matrix of the operator $\mathsf{P}_b^{(0,1)} \circ \mathsf{P}_d^{(1,0)}$ relative to the Haar basis $\{h_I\}_{I\in\mathcal{D}}$. A simple computation show that it has entries:

$$\begin{aligned}
G_{I,J} &= \left\langle \mathsf{P}_b^{(0,1)} \circ \mathsf{P}_d^{(1,0)} h_J, h_I \right\rangle_{L^2} = \left\langle \mathsf{P}_d^{(1,0)} h_J, \mathsf{P}_b^{(1,0)} h_I \right\rangle_{L^2} \\
&= \left\langle d_J\, h_J^1, b_I\, h_I^1 \right\rangle_{L^2} \\
&= \overline{b_I} d_J \frac{|I\cap J|}{|I|\,|J|} = \begin{cases} \overline{b_I} d_J \frac{1}{|I|} & \text{if} \quad J \subset I \\ \overline{b_I} d_J \frac{1}{|J|} & \text{if} \quad I \subset J \\ 0 & \text{if} \quad I\cap J = \emptyset. \end{cases}
\end{aligned}$$

Idea: Construct $\mathsf{T}_{b,d}^{(0,1,1,0)} : L^2_c(\mathcal{H}) \to L^2_c(\mathcal{H})$ that has the same Gram matrix as $\mathsf{P}_b^{(0,1)} \circ \mathsf{P}_d^{(1,0)}$, but with respect to the basis $\left\{\widetilde{\mathbf{1}}_{T(I)}\right\}_{I\in\mathcal{D}}$.

The Operator $\mathsf{T}_{b,d}^{(0,1,1,0)}$ and it Gram Matrix

For $\lambda \in \mathbb{R}$ and $a = \{a_I\}_{I\in\mathcal{D}}$ the multiplication operator $\mathcal{M}_a^\lambda$ is defined on basis elements $\widetilde{\mathbf{1}}_{T(K)}$ by

$$\mathcal{M}_a^\lambda \widetilde{\mathbf{1}}_{T(K)} \equiv a_K \left|K\right|^\lambda \widetilde{\mathbf{1}}_{T(K)}.$$

Define an operator $\mathsf{T}_{b,d}^{(0,1,1,0)}$ on $L_c^2(\mathcal{H})$ by

$$\mathsf{T}_{b,d}^{(0,1,1,0)} \equiv \mathcal{M}_{\overline{b}}^0 \left(\sum_{K\in\mathcal{D}} \widetilde{\mathbf{1}}_{T(K)} \otimes \widetilde{\mathbf{1}}_{Q(K)} \right) \mathcal{M}_d^{-1}.$$

Then the Gram matrix $\mathfrak{G}_{\mathsf{T}_{b,d}^{(0,1,1,0)}} = \left[G_{I,J}\right]_{I,J\in\mathcal{D}}$ of $\mathsf{T}_{b,d}^{(0,1,1,0)}$ relative to the basis $\left\{\widetilde{\mathbf{1}}_{T(I)}\right\}_{I\in\mathcal{D}}$ has entries

$$\begin{aligned} G_{I,J} &= \left\langle \mathsf{T}_{b,d}^{(0,1,1,0)} \widetilde{\mathbf{1}}_{T(J)}, \widetilde{\mathbf{1}}_{T(I)} \right\rangle_{L^2(\mathcal{H})} \\ &= \overline{b_I} d_J \sqrt{2} \frac{\left|Q(I) \cap T(J)\right|}{\left|I\right| \left|J\right|^2} = \frac{1}{\sqrt{2}} \begin{cases} \overline{b_I} d_J \frac{1}{|I|} & \text{if} \quad J \subset I \\ 0 & \text{if} \quad J \not\subset I. \end{cases} \end{aligned}$$

Connecting the Problem to a Two Weight Inequality

Up to an absolute constant, $\mathfrak{G}_{\mathsf{T}_{b,d}^{(0,1,1,0)}}$ matches $\mathfrak{G}_{\mathsf{P}_b^{(0,1)} \circ \mathsf{P}_d^{(1,0)}}$ in the lower triangle where $J \subset I$. So,

$$\left\| \mathsf{P}_b^{(0,1)} \circ \mathsf{P}_d^{(1,0)} \right\|_{L^2 \to L^2} \approx \left\| \mathsf{T}_{b,d}^{(0,1,1,0)} \right\|_{L^2(\mathcal{H}) \to L^2(\mathcal{H})} + \left\| \mathsf{T}_{d,b}^{(0,1,1,0)} \right\|_{L^2(\mathcal{H}) \to L^2(\mathcal{H})} .$$

The inequality we wish to characterize is

$$\left\| \mathcal{M}_b^0 \mathsf{U} \mathcal{M}_d^{-1} f \right\|_{L_c^2(\mathcal{H})} = \left\| \mathsf{T}_{b,d}^{(0,1,1,0)} f \right\|_{L_c^2(\mathcal{H})} \lesssim \|f\|_{L_c^2(\mathcal{H})} .$$

Define U on $L_c^2(\mathcal{H})$, where

$$\mathsf{U} \equiv \sum_{K \in \mathcal{D}} \widetilde{\mathbf{1}}_{T(K)} \otimes \widetilde{\mathbf{1}}_{Q(K)} .$$

For appropriate choice of weights σ and w on $\mathcal{H}$ the desired estimate is simply:

$$\|\mathsf{U}(\sigma k)\|_{L_c^2(\mathcal{H};w)} \lesssim \|k\|_{L_c^2(\mathcal{H};\sigma)} .$$

A Two Weight Theorem for Positive Operators

Theorem (S. Pott, E. Sawyer, M. Reguera-Rodriguez, BDW)

Let w and σ be non-negative weights on $\mathcal{H}$. Then

$$\mathsf{U}(\sigma\cdot) : L^2(\mathcal{H};\sigma) \to L^2(\mathcal{H};w)$$

is bounded if and only if the following testing condition holds:

$$\left\| \mathbf{1}_{Q(I)} \mathsf{U}\left(\sigma \mathbf{1}_{Q(I)}\right)\right\|_{L^2(\mathcal{H};w)}^2 \leq C_0^2 \left\| \mathbf{1}_{Q(I)} \right\|_{L^2(\mathcal{H};\sigma)}^2 .$$

- The proof of this Theorem is a translation of Sawyer's proof strategy for two weight inequalities for positive operators.
- Choosing $w \equiv \sum_{I\in\mathcal{D}} |b_I|^2 \mathbf{1}_{T(I)}$ and $\sigma \equiv \sum_{I\in\mathcal{D}} \frac{|d_I|^2}{|I|^2} \mathbf{1}_{T(I)}$ (and unraveling the definitions) gives the forward testing condition.
- Appropriate choice of w and σ will then provide the backward testing condition when studying $T_{d,b}^{(0,1,1,0)}$.

The Gram Matrix of $\mathsf{P}_b^{(0,1)} \circ \mathsf{P}_d^{(0,0)}$

Let $\mathfrak{G}_{\mathsf{P}_b^{(0,1)} \circ \mathsf{P}_d^{(0,0)}} = [G_{I,J}]_{I,J \in \mathcal{D}}$ be the Gram matrix of the operator $\mathsf{P}_b^{(0,1)} \circ \mathsf{P}_d^{(0,0)}$ relative to the Haar basis $\{h_I\}_{I \in \mathcal{D}}$. A simple computation shows its entries are:

$$
\begin{aligned}
G_{I,J} &= \left\langle \mathsf{P}_b^{(0,1)} \circ \mathsf{P}_d^{(0,0)} h_J, h_I \right\rangle_{L^2} = \left\langle \mathsf{P}_d^{(0,0)} h_J, \mathsf{P}_b^{(1,0)} h_I \right\rangle_{L^2} \\
&= \left\langle d_J h_J, b_I h_I^1 \right\rangle_{L^2} \\
&= \overline{b_I} d_J \widehat{h_I^1}(J) = \begin{cases} \overline{b_I} d_J \frac{-1}{\sqrt{|J|}} & \text{if} \quad I \subset J_- \\ \overline{b_I} d_J \frac{1}{\sqrt{|J|}} & \text{if} \quad I \subset J_+ \\ 0 & \text{if} \quad J \subset I \text{ or } I \cap J = \emptyset. \end{cases}
\end{aligned}
$$

Idea: Construct $\mathsf{T}_{b,d}^{(0,1,0,0)} : L_c^2(\mathcal{H}) \to L_c^2(\mathcal{H})$ that has the same Gram matrix as $\mathsf{P}_b^{(0,1)} \circ \mathsf{P}_d^{(0,0)}$, but with respect to the basis $\left\{ \widetilde{\mathbf{1}}_{T(I)} \right\}_{I \in \mathcal{D}}$.

The Operator $\mathsf{T}_{b,d}^{(0,1,0,0)}$

Now consider the operator $\mathsf{T}_{b,d}^{(0,1,0,0)}$ defined by

$$\mathsf{T}_{b,d}^{(0,1,0,0)} \equiv \mathcal{M}_{\overline{b}}^{-1} \left(\sum_{K \in \mathcal{D}} \widetilde{\mathbf{1}}_{Q_{\pm}(K)} \otimes \widetilde{\mathbf{1}}_{T(K)} \right) \mathcal{M}_{d}^{\frac{1}{2}}.$$

Here

$$\mathbf{1}_{Q_{\pm}(K)} \equiv - \sum_{L \subset K_-} \mathbf{1}_{T(L)} + \sum_{L \subset K_+} \mathbf{1}_{T(L)}.$$

A straightforward computation shows

$$\begin{aligned} \left\| \mathbf{1}_{Q_{\pm}(K)} \right\|_{L^2(\mathcal{H})} &= \frac{|K|}{2}; \\ \mathcal{M}_a^{\lambda} \mathbf{1}_{Q_{\pm}(K)} &= - \sum_{L \subset K_-} a_L \left| L \right|^{\lambda} \mathbf{1}_{T(L)} + \sum_{L \subset K_+} a_L \left| L \right|^{\lambda} \mathbf{1}_{T(L)}. \end{aligned}$$

The Gram Matrix for the Operator $\mathsf{T}_{b,d}^{(0,1,0,0)}$

The Gram matrix $\mathfrak{G}_{\mathsf{T}_{b,d}^{(0,1,0,0)}} = [G_{I,J}]_{I,J\in\mathcal{D}}$ of $\mathsf{T}_{b,d}^{(0,1,0,0)}$ relative to the basis $\left\{\widetilde{\mathbf{1}}_{T(I)}\right\}_{I\in\mathcal{D}}$ then has entries given by

$$\begin{aligned} G_{I,J} &= \left\langle \mathsf{T}_{b,d}^{(0,1,0,0)} \widetilde{\mathbf{1}}_{T(J)}, \widetilde{\mathbf{1}}_{T(I)} \right\rangle_{L^2(\mathcal{H})} \\ &= \sqrt{2} \begin{cases} -\overline{b_I} d_J \, |J|^{-\frac{1}{2}} & \text{if} \quad I \subset J_- \\ \overline{b_I} d_J \, |J|^{-\frac{1}{2}} & \text{if} \quad I \subset J_+ \\ 0 & \text{if} \quad J \subset I \text{ or } I \cap J = \emptyset. \end{cases} \end{aligned}$$

Thus, up to an absolute constant, $\mathfrak{G}_{\mathsf{T}_{b,d}^{(0,1,0,0)}} = \mathfrak{G}_{\mathsf{P}_b^{(0,1)} \circ \mathsf{P}_d^{(0,0)}}$, and so

$$\left\| \mathsf{P}_b^{(0,1)} \circ \mathsf{P}_d^{(0,0)} \right\|_{L^2 \to L^2} \approx \left\| \mathsf{T}_{b,d}^{(0,1,0,0)} \right\|_{L^2(\mathcal{H}) \to L^2(\mathcal{H})} .$$

Connecting to a Two Weight Inequality

The inequality we wish to characterize is:

$$\left\| \mathcal{M}_{\overline{b}}^{-1} \mathsf{U} \mathcal{M}_{d}^{\frac{1}{2}} f \right\|_{L_c^2(\mathcal{H})} = \left\| \mathsf{T}_{b,d}^{(0,1,0,0)} f \right\|_{L_c^2(\mathcal{H})} \lesssim \|f\|_{L_c^2(\mathcal{H})} .$$

Where the operator U on $L^2(\mathcal{H})$ is defined by

$$\mathsf{U} \equiv \sum_{K \in \mathcal{D}} \widetilde{\mathbf{1}}_{Q_{\pm}(K)} \otimes \widetilde{\mathbf{1}}_{T(K)} .$$

One sees that the inequality to be characterized is equivalent to:

$$\| \mathsf{U} (\mu g) \|_{L_c^2(\mathcal{H};\nu)} \lesssim \|g\|_{L_c^2(\mathcal{H};\mu)} ,$$

where the weights μ and ν are given by

$$\begin{aligned} \nu &\equiv \sum_{I \in \mathcal{D}} |b_I|^2 |I|^{-2} \mathbf{1}_{T(I)} \\ \mu &\equiv \sum_{I \in \mathcal{D}} |d_I|^{-2} |I|^{-1} \mathbf{1}_{T(I)} . \end{aligned}$$

Theorem (S. Pott, E. Sawyer, M. Reguera-Rodriguez, BDW)

Suppose that μ and ν are positive measures on $\mathcal{H}$ that are constant on tiles, i.e., $\mu \equiv \sum_{I\in\mathcal{D}} \mu_I \mathbf{1}_{T(I)}$, $\nu \equiv \sum_{I\in\mathcal{D}} \nu_I \mathbf{1}_{T(I)}$. Then

$$\mathsf{U}(\mu\cdot) : L^2_c(\mathcal{H};\mu) \to L^2_c(\mathcal{H};\nu)$$

if and only if both

$$\begin{aligned}
\left\|\mathsf{U}\left(\mu\mathbf{1}_{T(I)}\right)\right\|_{L^2_c(\mathcal{H};\nu)} &\leq C_1 \left\|\mathbf{1}_{T(I)}\right\|_{L^2_c(\mathcal{H};\mu)} = \sqrt{\mu(T(I))},\\
\left\|\mathbf{1}_{Q(I)}\mathsf{U}^*\left(\nu\mathbf{1}_{Q(I)}\right)\right\|_{L^2_c(\mathcal{H};\mu)} &\leq C_2 \left\|\mathbf{1}_{Q(I)}\right\|_{L^2_c(\mathcal{H};\nu)} = \sqrt{\nu(Q(I))},
\end{aligned}$$

hold for all $I \in \mathcal{D}$. Moreover, we have that

$$\|\mathsf{U}\|_{L^2_c(\mathcal{H};\mu)\to L^2_c(\mathcal{H};\nu)} \approx C_1 + C_2$$

where C_1 and C_2 are the best constants appearing above.

An Application: Linear Bound for Hilbert Transform

- For a weight w, i.e., a positive locally integrable function on $\mathbb{R}$, let $L^2(w) \equiv L^2(\mathbb{R}; w)$.
- A weight belongs to A_2 if: $[w]_{A_2} \equiv \sup_I \langle w \rangle_I \langle w^{-1} \rangle_I < +\infty$.
- The Hilbert transform is the operator: $H(f)(x) \equiv \text{p.v.} \int_{\mathbb{R}} \frac{f(y)}{y-x} dy$.

Theorem (Petermichl)

Let $w \in A_2$. Then $\|H\|_{L^2(w) \to L^2(w)} \lesssim [w]_{A_2}$, and the linear growth is optimal.

- $\|T\|_{L^2(w) \to L^2(w)} = \left\| M_{w^{\frac{1}{2}}} T M_{w^{-\frac{1}{2}}} \right\|_{L^2 \to L^2}$;
- H is the average of dyadic shifts Ш;
- $M_{w^{\frac{1}{2}}} Ш M_{w^{-\frac{1}{2}}}$ can be written as a sum of nine compositions of paraproducts; Some of which are amenable to the Theorems above.
- However, each term can be shown to have norm no worse than $[w]_{A_2}$.

An Open Question

Unfortunately, the methods described do not appear to work to handle type $(0,1,0,1)$ compositions. However, the following question is of interest:

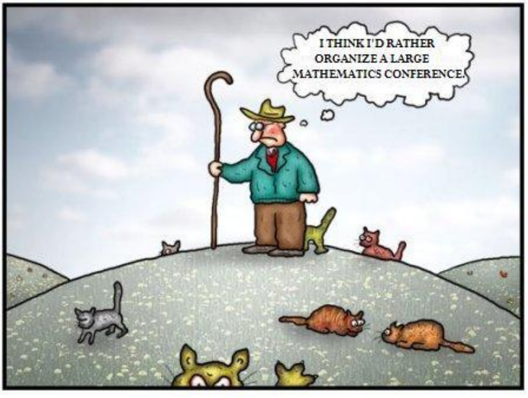
Question

For each $I \in \mathcal{D}$ determine function $F_I, B_I \in L^2$ of norm 1 such that $\mathsf{P}_b^{(0,1)} \circ \mathsf{P}_d^{(0,1)}$ is bounded on L^2 if and only if

$$\begin{aligned} \left\|\mathsf{P}_b^{(0,1)} \circ \mathsf{P}_d^{(0,1)} F_I\right\|_{L^2} &\leq C_1 \quad \forall I \in \mathcal{D}; \\ \left\|\mathsf{P}_d^{(1,0)} \circ \mathsf{P}_b^{(1,0)} B_I\right\|_{L^2} &\leq C_2 \quad \forall I \in \mathcal{D}. \end{aligned}$$

Moreover, we will have

$$\left\|\mathsf{P}_b^{(0,1)} \circ \mathsf{P}_d^{(0,1)}\right\|_{L^2 \to L^2} \approx C_1 + C_2.$$



The daydreams of cat herders

(Modified from the Original Dr. Fun Comic)

Thanks to Dechao for Organizing the Meeting!

Thank You!