Carleson Measures for Hilbert Spaces of Analytic Functions

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Definition ($\mathcal{H}$-Carleson Measure)

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Question

Give a 'geometric' and 'testable' characterization of the $\mathcal{H}$-Carleson measures.

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Here $T(I_\lambda)$ is the 'tent' over the set I_λ in the boundary $\partial\Omega$.

Reasons to Care about Carleson Measures

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- I am an analyst that cares mores about one complex variable, inner functions, Carleson measures, and their interaction.

  ▸ Characterization of Carleson Measures for the Model Space K_ϑ on $\mathbb{D}$

Besov-Sobolev Spaces

- The space $B_2^\sigma(\mathbb{B}_n)$ is the collection of holomorphic functions f on the unit ball $\mathbb{B}_n = \{z \in \mathbb{C}^n : |z| < 1\}$ such that

$$\left\{ \sum_{k=0}^{m-1} \left| f^{(k)}(0) \right|^2 + \int_{\mathbb{B}_n} \left| \left(1 - |z|^2\right)^{m+\sigma} f^{(m)}(z) \right|^2 d\lambda_n(z) \right\}^{\frac{1}{2}} < \infty,$$

where $d\lambda_n(z) = \left(1 - |z|^2\right)^{-n-1} dV(z)$ is the invariant measure on $\mathbb{B}_n$ and $m + \sigma > \frac{n}{2}$.

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Question (Main Problem: Characterization in the Difficult Range)

Characterize the Carleson measures when $\frac{1}{2} < \sigma < \frac{n}{2}$.

Operator Theoretic Characterization of Carleson Measures

A measure μ is Carleson exactly if the inclusion map ι from $\mathcal{H}$ to $L^2(\Omega;\mu)$ is bounded, or

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A simple functional analysis argument lets one recast this in an equivalent way:

Proposition (Arcozzi, Rochberg, and Sawyer)

A measure μ is a $\mathcal{H}$-Carleson measure if and only if the linear map

$$T(f)(z) = \int_\Omega \operatorname{Re} K_x(z) \, f(x) d\mu(x)$$

is bounded on $L^2(\Omega;\mu)$.

Connections to Calderón-Zygmund Operators

When we apply this proposition to the spaces $B_2^\sigma(\mathbb{B}_n)$ this suggests that we study the operator

$$T_{\mu,2\sigma}(f)(z) = \int_{\mathbb{B}_n} \text{Re}\left(\frac{1}{(1-\overline{w}z)^{2\sigma}}\right) f(w)d\mu(w) : L^2(\mathbb{B}_n;\mu) \to L^2(\mathbb{B}_n;\mu)$$

and find some conditions that will let us determine when it is bounded.

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- Idea: Try to use the T(1)-Theorem from harmonic analysis to characterize the boundedness of

$$T_{\mu,2\sigma} : L^2(\mathbb{B}_n;\mu) \to L^2(\mathbb{B}_n;\mu).$$

Danger: Proof will Fail without Coordination!

Calderón–Zygmund Estimates for $T_{\mu,2\sigma}$

If we define

$$\Delta(z,w) := \begin{cases} \left||z|-|w|\right| + \left|1 - \frac{z\overline{w}}{|z||w|}\right| & : \quad z,w \in \mathbb{B}_n \setminus \{0\} \\ |z|+|w| & : \quad \text{otherwise.} \end{cases}$$

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A computation demonstrates that the kernel of $T_{\mu,2\sigma}$ satisfies the following estimates:

$$|K_{2\sigma}(z,w)| \lesssim \frac{1}{\Delta(z,w)^{2\sigma}} \quad \forall z,w \in \mathbb{B}_n;$$

If $\Delta(\zeta,w) < \frac{1}{2}\Delta(z,w)$ then

$$|K_{2\sigma}(\zeta,w) - K_{2\sigma}(z,w)| \lesssim \frac{\Delta(\zeta,w)^{1/2}}{\Delta(z,w)^{2\sigma+1/2}}.$$

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  and this is exactly the phenomenon that will save us!
- This places us in the setting of non-homogeneous harmonic analysis as developed by Nazarov, Treil, and Volberg. We have an operator with a Calderón-Zygmund kernel satisfying estimates of order 2σ, a measure μ of order 2σ, and are interested in $L^2(\mathbb{B}_n;\mu) \to L^2(\mathbb{B}_n;\mu)$ bounds.

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More precisely, for $m \leq d$ we are interested in Calderón-Zygmund kernels that satisfy the following estimates:

$$|k(x,y)| \leq \frac{C_{CZ}}{|x-y|^m},$$

and

$$|k(y,x) - k(y,x')| + |k(x,y) - k(x',y)| \leq C_{CZ} \frac{|x-x'|^\tau}{|x-y|^{m+\tau}}$$

provided that $|x - x'| \leq \frac{1}{2}|x - y|$, with some (fixed) $0 < \tau \leq 1$ and $0 < C_{CZ} < \infty$.

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Additionally the kernels will have the following property

$$|k(x,y)| \leq \frac{1}{\max(d(x)^m, d(y)^m)},$$

where $d(x) := \text{dist}(x, \mathbb{R}^d \setminus H)$ and H being an open set in $\mathbb{R}^d$.

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Once the kernel has been defined, then we say that a $L^2(\mathbb{R}^d; \mu)$ bounded operator is a Calderón-Zygmund operator with kernel k if,

$$T_{\mu,m} f(x) = \int_{\mathbb{R}^d} k(x,y) f(y) d\mu(y) \quad \forall x \notin \text{supp} f .$$

T(1)-Theorem for Bergman-Type Operators

Theorem (T(1)-Theorem for Bergman-Type Operators, Volberg and W., Amer. J. Math., 134 (2012))

Let $k(x,y)$ be a Calderón-Zygmund kernel of order m on $X \subset \mathbb{R}^d$, $m \leq d$ with Calderón-Zygmund constants C_{CZ} and τ. Let μ be a probability measure with compact support in X and all balls such that $\mu(B_r(x)) > r^m$ lie in an open set H. Let also

$$|k(x,y)| \leq \frac{1}{\max(d(x)^m, d(y)^m)},$$

where $d(x) := \operatorname{dist}(x, \mathbb{R}^d \setminus H)$. Finally, suppose also that:

$$\| T_{\mu,m}\chi_Q\|^2_{L^2(\mathbb{R}^d;\mu)} \leq A\,\mu(Q)\,,\ \| T^*_{\mu,m}\chi_Q\|^2_{L^2(\mathbb{R}^d;\mu)} \leq A\,\mu(Q)\,.$$

Then $\| T_{\mu,m}\|_{L^2(\mathbb{R}^d;\mu)\to L^2(\mathbb{R}^d;\mu)} \leq C(A,m,d,\tau)$.

Main Results

Theorem (Characterization of Carleson Measures for $B^2_\sigma(\mathbb{B}_n)$, Volberg and W., Amer. J. Math., 134 (2012))

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Above, the sets B_Δ are balls measured with respect to the metric Δ and the set Q is a "cube" defined with respect to the metric Δ.

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  - Follow the proof strategy for the T(1) theorem in the context at hand. Technical but well established path (safe route!).
- It is possible to show that the T(1) condition reduces to the simpler conditions in certain cases.
- An alternate proof of this Theorem was later given by Hytönen and Martikainen. Their proof used a non-homogeneous T(b)-Theorem on metric spaces!

Safe Passage to the End!

Return to Beginning · Conclusion · Details

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Question (Carleson Measure Problem for K_ϑ)

Geometrically characterize the Carleson measures for K_ϑ:

$$\int_{\overline{\mathbb{D}}} |f(z)|^2 \, d\mu(z) \leq C(\mu)^2 \, \|f\|_{K_\vartheta}^2 \quad \forall f \in K_\vartheta.$$

Carleson Measures for K_ϑ

We always have the necessary condition:

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- Nazarov and Volberg proved the obvious necessary condition is not sufficient for μ to be a K_ϑ-Carleson measure.

The Two-Weight Cauchy Transform

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Two-Weight Inequality for the Cauchy Transform

Theorem (Lacey, Sawyer, Shen, Uriarte-Tuero, W.)

Let σ be a weight on $\mathbb{T}$ and τ a weight on $\overline{\mathbb{D}}$. The inequality below holds, for some finite positive $\mathscr{C}$,

$$\|\mathsf{C}(\sigma f)\|_{L^2(\overline{\mathbb{D}};\tau)} \leq \mathscr{C}\|f\|_{L^2(\mathbb{T};\sigma)},$$

if and only if these constants are finite:

$$\sigma(\mathbb{T}) \cdot \tau(\overline{\mathbb{D}}) + \sup_{z\in\mathbb{D}}\{\mathsf{P}(\sigma\mathbf{1}_{\mathbb{T}\setminus I})(z)\mathsf{P}\tau(z) + \mathsf{P}\sigma(z)\mathsf{P}(\tau\mathbf{1}_{\overline{\mathbb{D}}\setminus B_I})(z)\} \equiv \mathscr{A}_2,$$

$$\sup_I \sigma(I)^{-1}\int_{B_I}|\mathsf{C}_\sigma\mathbf{1}_I(z)|^2\tau(dA(z)) \equiv \mathscr{T}^2,$$

$$\sup_I \tau(B_I)^{-1}\int_I|\mathsf{C}_\tau^*\mathbf{1}_{B_I}(w)|^2\sigma(dw) \equiv \mathscr{T}^2.$$

Finally, we have $\mathscr{C} \simeq \mathscr{A}_2^{1/2} + \mathscr{T}$.

Danger: Technical Obstructions Exist!

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Write the coordinates of this operator as $(\mathrm{R}^1, \mathrm{R}^2)$. The second coordinate R^2 is the Poisson transform P. The Cauchy transform is

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Question

Let σ denote a weight on $\mathbb{R}$ and τ denote a measure on the upper half plane $\mathbb{R}^2_+$. Find necessary and sufficient conditions on the pair of measures σ and τ so that the estimate below holds:

$$\|R_\sigma(f)\|_{L^2(\mathbb{R}^2_+;\tau)} = \|\mathsf{R}(\sigma f)\|_{L^2(\mathbb{R}^2_+;\tau)} \leq \mathscr{N}\,\|f\|_{L^2(\mathbb{R};\sigma)}.$$

Two Weight for Cauchy/Riesz Transforms

Theorem (Lacey, Sawyer, Shen, Uriarte-Tuero, W.)

Let σ be a weight on $\mathbb{R}$ and τ a weight on the closed upper half-plane $\mathbb{R}^2_+$. Then $\|R_\sigma(f)\|_{L^2(\mathbb{R}^2_+;\tau)} \leq \mathscr{N}\,\|f\|_{L^2(\mathbb{R};\sigma)}$ if and only if for a finite positive constant $\mathscr{A}_2$ and $\mathscr{T}$,

$$\frac{\tau(Q_I)}{|I|} \times \int_{\mathbb{R}\setminus I} \frac{|I|}{(|I| + \operatorname{dist}(t, I))^2}\,\sigma(dt) \leq \mathscr{A}_2,$$

$$\frac{\sigma(I)}{|I|} \times \int_{\mathbb{R}^2_+\setminus Q_I} \frac{|I|}{(|I| + \operatorname{dist}(x, Q_I))^2}\,\tau(dx) \leq \mathscr{A}_2,$$

$$\int_{Q_I} |\mathrm{R}_\sigma \mathbf{1}_I(x)|^2\,\tau(dx) \leq \mathscr{T}^2 \sigma(I) \quad \text{and} \quad \int_I |\mathrm{R}^*_\tau \mathbf{1}_{Q_I}(t)|^2\,\sigma(dt) \leq \mathscr{T}^2 \tau(Q_I).$$

Moreover, $\mathscr{N} \simeq \mathscr{A}_2^{1/2} + \mathscr{T}$.

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$$\sup_I \frac{1}{\tau(Q_I)} \int_I |\mathrm{R}_\tau^* \mathbf{1}_{Q_I}(t)|^2\, \sigma(dt) \leq \mathscr{T}^2.$$

- Majority of efforts go into showing sufficiency of these conditions.

Necessary Conditions: Testing on Intervals/Cubes

Assuming that the Riesz transforms are bounded we have:

$$\|R_\sigma(f)\|_{L^2(\mathbb{R}^2_+;\tau)} = \|\mathrm{R}\,(\sigma f)\|_{L^2(\mathbb{R}^2_+;\tau)} \leq \mathscr{N}\,\|f\|_{L^2(\mathbb{R};\sigma)}\,.$$

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A simple duality argument to show that:

$$\left\| R_\tau^*(f) \right\|_{L^2(\mathbb{R};\sigma)} = \left\| \mathrm{R}\left(\tau f\right) \right\|_{L^2(\mathbb{R};\sigma)} \leq \mathscr{N} \left\| f \right\|_{L^2(\mathbb{R}^2_+;\tau)} .$$

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This implies:

$$\begin{aligned}
\int_{Q_I} |\mathrm{R}_\sigma \mathbf{1}_I(x)|^2\,\tau(dx) &\leq \|R_\sigma(1_I)\|^2_{L^2(\mathbb{R}^2_+;\tau)} \leq \mathscr{N}^2\,\|\mathbf{1}_I\|^2_{L^2(\mathbb{R};\sigma)} = \mathscr{N}^2\sigma(I).\\
\int_I |\mathrm{R}^*_\tau \mathbf{1}_{Q_I}(t)|^2\,\sigma(dt) &\leq \|R^*_\tau(1_{Q_I})\|^2_{L^2(\mathbb{R};\sigma)} \leq \mathscr{N}^2\,\|\mathbf{1}_{Q_I}\|^2_{L^2(\mathbb{R}^2_+;\tau)} = \mathscr{N}^2\tau(Q_I).
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\end{aligned}$$

Which gives that $\mathscr{T} \leq \mathscr{N}$.

Necessary Conditions: Two Weight A_2

This is also a well-known argument. Both directions are similar and resort to testing on a function like:

$$p_I(x)^2 = \frac{|I|}{(|I| + \mathrm{dist}(x, I))^2}$$

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$$\begin{aligned} \frac{\tau(Q_I)}{|I|} \left(\int_{\mathbb{R}\setminus I} \frac{|I|}{(|I| + \mathrm{dist}(t, I))^2} \, \sigma(dt) \right)^2 &\leq \|\mathrm{R}(\sigma p_I)\|^2_{L^2(\mathbb{R}^2_+;\tau)} \\ &\lesssim \mathscr{N}^2 \, \|p_I\|^2_{L^2(\mathbb{R}\setminus I;\sigma)} . \end{aligned}$$

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Main Ideas behind the Proof

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$$\begin{aligned}\langle \mathsf{R}_\tau^* \varphi, h_J^\sigma \rangle_\sigma &= \iint_{\mathbb{R}^2_+ \setminus Q_J} \int_J \varphi(x) h_J^\sigma(t) \frac{x-t}{|x-t|^2}\, \sigma(dt)\, \tau(dx) \\ &= \iint_{\mathbb{R}^2_+ \setminus Q_J} \int_J \varphi(x) h_J^\sigma(t) \left(\frac{x-t}{|x-t|^2} - \frac{x-t_J}{|x-t_J|^2} \right) \sigma(dt)\, \tau(dx) \\ &\simeq \mathsf{T}_\tau \varphi(x_{Q_J}) \cdot \left\langle \frac{t}{|J|}, h_{J'}^\sigma \right\rangle_\sigma .\end{aligned}$$

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Lemma

For all intervals I_0 and partitions $\mathcal{I}$ of I_0 into dyadic intervals,

$$\sum_{I \in \mathcal{I}} \sum_{K \in \mathcal{W} I} \mathsf{T}_\tau (Q_{I_0} \setminus Q_K)(x_{Q_K})^2 \left(\frac{1}{\sigma(I)} \sum_{\substack{J \,:\, J \subset I \\ J \text{ is good}}} \left\langle \frac{t}{|I|}, h_J^\sigma \right\rangle_\sigma^2 \right) \sigma(K) \lesssim \mathscr{R}^2 \tau(Q_{I_0}).$$

Connecting the Cauchy Transform to Carleson Measures

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- For the inner function ϑ and measure μ, define a new measure $\nu_{\vartheta,\mu} \equiv |1-\vartheta|^2 \mu$.

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Lemma (Nazarov, Volberg)

A measure μ is a Carleson measure for K_ϑ if and only if $\mathrm{C} : L^2(\mathbb{T};\sigma) \to L^2(\overline{\mathbb{D}};\nu_{\vartheta,\mu})$ is bounded.

Characterization of Carleson Measures for K_ϑ

Theorem (Lacey, Sawyer, Shen, Uriarte-Tuero, W.)

Let μ be a non-negative Borel measure supported on $\overline{\mathbb{D}}$ and let ϑ be an inner function on $\mathbb{D}$ with Clark measure σ. Set $\nu_{\mu,\vartheta} = |1-\vartheta|^2\mu$. The following are equivalent:

(i) *μ is a Carleson measure for K_ϑ, namely,*

$$\int_{\overline{\mathbb{D}}} |f(z)|^2 \, d\mu(z) \leq C(\mu)^2 \, \|f\|_{K_\vartheta}^2 \quad \forall f \in K_\vartheta;$$

(ii) *The Cauchy transform C is a bounded map between $L^2(\mathbb{T};\sigma)$ and $L^2(\overline{\mathbb{D}};\nu_{\mu,\vartheta})$, i.e., $\mathsf{C} : L^2(\mathbb{T};\sigma) \to L^2(\overline{\mathbb{D}};\nu_{\vartheta,\mu})$ is bounded;*

(iii) *The three conditions in the above theorem hold for the pair of measures σ and $\nu_{\mu,\vartheta}$. Moreover,*

$$C(\mu) \simeq \|\mathsf{C}\|_{L^2(\mathbb{T};\sigma)\to L^2(\overline{\mathbb{D}};\nu_{\vartheta,\mu})} \simeq \mathscr{A}_2^{1/2} + \mathscr{T}.$$

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  - Follow the proof strategy as initiated by Nazarov, Treil, and Volberg. Use required modifications developed by Lacey, Sawyer, Shen, Uriarte-Tuero. Technical but established path (safe route!).
- It is possible to show that a similar characterization exists for d-dimensional Riesz transforms in $\mathbb{R}^n$ provided the weights satisfy some restrictions (e.g., σ and τ are doubling, one weight supported on a line). Full characterization is open still.

Carleson Measures and Composition Operators

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To the function φ and weight τ we associate the pullback measure τ_φ defined as a measure on $\overline{\mathbb{D}}$, as $\tau_\varphi(E) \equiv \tau(\varphi^{-1}(E))$. Then

$$\|C_\varphi f\|^2_{H^2_\tau} = \int_{\overline{\mathbb{D}}} |f \circ \varphi(z)|^2 \, \tau(dA(z)) = \int_{\overline{\mathbb{D}}} |f(z)|^2 \, \tau_\varphi(dA(z)).$$

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Behavior of the composition operator $C_\varphi : K_\vartheta \to H^2_\tau(\mathbb{D})$ is equivalent to corresponding behavior of τ_φ as a Carleson measure for K_ϑ.

Bounded, Compact, Essential Norm of Composition Operators

Theorem (Lacey, Sawyer, Shen, Uriarte-Tuero, W.)

Let ϑ be an inner function. Let $\varphi : \mathbb{D} \to \mathbb{D}$ be analytic and let τ_φ denote the pullback measure associated to φ. The following are equivalent:

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(ii) τ_φ *is a Carleson measure for* K_ϑ*, namely,*

$$\int_{\mathbb{D}} |f(z)|^2 \, \tau_\varphi(dA(z)) \leq C(\tau_\varphi)^2 \, \|f\|^2_{K_\vartheta} \quad \forall f \in K_\vartheta;$$

(iii) *The testing and* A_2 *conditions hold for the pair of weights* σ *on* $\mathbb{T}$ *and* $\nu_{\tau_\varphi,\vartheta} = |1-\vartheta|^2 \tau_\varphi$ *on* $\mathbb{D}$*.*

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Compactness and essential norm can also be obtained from this result.

Safe Passage to the End!

Return to Beginning
Conclusion
Details

Commonalities of the Proof

- In both situations we are left studying the boundedness of an operator $T : L^2(u) \to L^2(v)$ (with the possibility that $u = v$).

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  - In the case of the unit ball, the geometry dictates the grids.
  - In the model space case, the grid on $\mathbb{R}$ influences the construction of the grid in the upper half plane.
- Define expectation operators Δ_Q (Haar function on Q) and Λ (average on Q^0), then we have for every $f \in L^2(u)$

$$
\begin{aligned}
f &= \Lambda f + \sum_{Q \in \mathcal{D}_1} \Delta_Q f \\
\|f\|_{L^2(u)}^2 &= \|\Lambda f\|_{L^2(u)}^2 + \sum_{Q \in \mathcal{D}_1} \|\Delta_Q f\|_{L^2(u)}^2 .
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Good and Bad Decomposition

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- Decomposition of f and g into good and bad parts:

$$f = f_{good} + f_{bad}\,, \text{ where } f_{good} = \Lambda f + \sum_{Q \in \mathcal{D}_1 \cap \mathcal{G}_1} \Delta_Q f$$

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- The probability that a cube is bad is small: $\mathbb{P}\{Q \text{ is bad}\} \leq \delta^2$ and $\mathbb{E}(\|f_{bad}\|_{L^2(u)}) \leq \delta \|f\|_{L^2(u)}$.

Good and Bad Decomposition

- Define good and bad cubes. Heuristically, a cube $Q \in \mathcal{D}_1$ is *bad* if there is a cube $R \in \mathcal{D}_2$ of bigger size and Q is close to the boundary of R. More precisely, fix $0 < \delta < 1$ and $r \in \mathbb{N}$. $Q \in \mathcal{D}_1$ is said to be (δ, r)-*bad* if there is $R \in \mathcal{D}_2$ such that $|R| > 2^r |Q|$ and $\mathrm{dist}(Q, \partial R) < |Q|^{\delta} |R|^{1-\delta}$.
- Decomposition of f and g into good and bad parts:

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- Similar Statements for g hold as well.

Reduction to Controlling The Good Part

- Using the decomposition above, we have

$$\langle Tf, g\rangle_{L^2(v)} = \langle Tf_{good}, g_{good}\rangle_{L^2(v)} + R(f,g)$$

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- This will then give $\|T\|_{L^2(u)\to L^2(v)} \leq 2C$.

The fork in the road...

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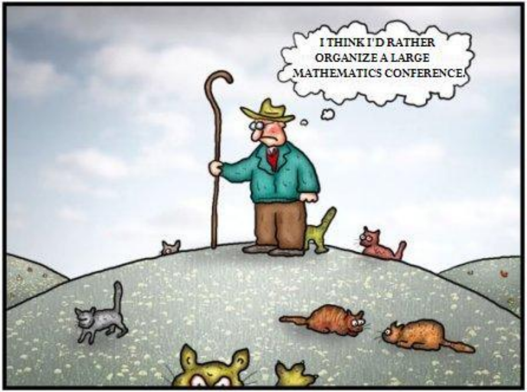
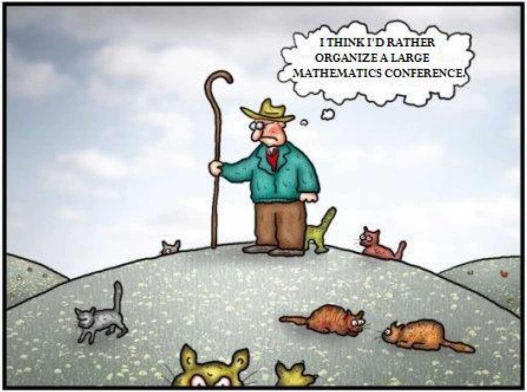
- Reduce to positive operators and use the testing conditions to control terms.
- Certain terms are amenable to direct estimates of the kernel, reducing to positive operators.
  - For the Cauchy transform follow the proof strategy for the Hilbert transform.
  - For the Besov-Sobolev projection follow more standard $T1$ proof strategies.

◂ Conclusion



The daydreams of cat herders



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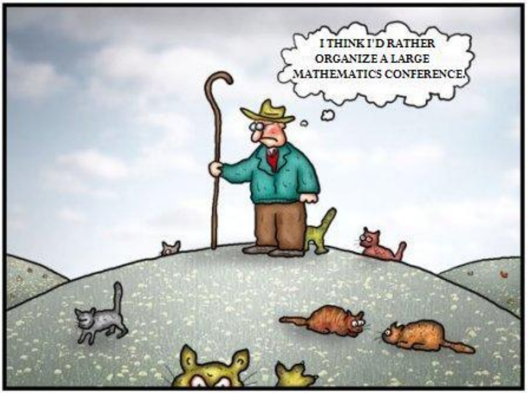
(Modified from the Original Dr. Fun Comic)



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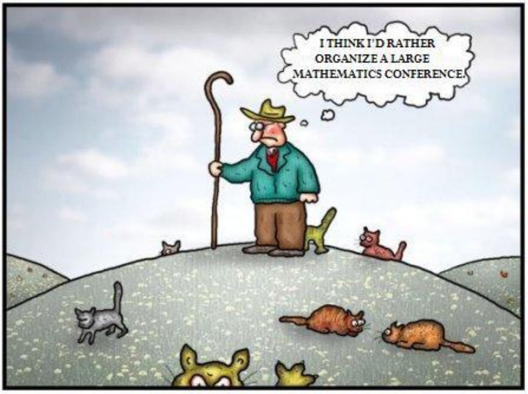
Thanks for arranging the Meeting!



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Thank You!

Comments & Questions