Function Theory meets Operator Theory: The Corona Problem and Bilinear Forms

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Where Did the Name Come From?

The Corona Problem for $H^\infty(\mathbb{D})$

The Banach algebra $H^\infty(\mathbb{D})$ is the collection of all analytic functions on the disc such that

$$\|f\|_{H^\infty(\mathbb{D})} := \sup_{z\in\mathbb{D}} |f(z)| < \infty.$$

Let $\varphi : H^\infty(\mathbb{D}) \to \mathbb{C}$ be a multiplicative linear functional. Namely,

$$\varphi(fg) = \varphi(f)\varphi(g) \quad \text{and} \quad \varphi(f+g) = \varphi(f) + \varphi(g).$$

It's an easy exercise to show that for any multiplicative linear functional

$$\sup_{f\in H^\infty(\mathbb{D})} |\varphi(f)| \leq \|f\|_{H^\infty(\mathbb{D})}.$$

To each $z \in \mathbb{D}$ we can associate a multiplicative linear functional on $H^\infty(\mathbb{D})$:

$$\varphi_z(f) := f(z) \quad \text{(point evaluation at } z\text{)}.$$

Where Did the Name Come From?

The Corona Problem for $H^\infty(\mathbb{D})$

Every non-trivial multiplicative linear functional φ determines a maximal (proper) ideal of $H^\infty(\mathbb{D})$: $\ker\varphi = \{f \in H^\infty(\mathbb{D}) : \varphi(f) = 0\}$.

Conversely, if M is a maximal (proper) ideal of $H^\infty(\mathbb{D})$ then $M = \ker\varphi$ for some non-trivial multiplicative linear functional.

The maximal ideal space of $H^\infty(\mathbb{D})$, $\mathcal{M}_{H^\infty(\mathbb{D})}$, is the collection of all multiplicative linear functionals φ.

We then have that the maximal ideal space is contained in the unit ball of the dual space $H^\infty(\mathbb{D})$. If we put the weak-$*$ topology on this space then $\mathcal{M}_{H^\infty(\mathbb{D})}$ is a compact Hausdorff space.

The proceeding discussion then shows that $\mathbb{D} \subset \mathcal{M}_{H^\infty(\mathbb{D})}$.

Where Did the Name Come From?

The Corona Problem for $H^\infty(\mathbb{D})$

One then defines the Corona of $H^\infty(\mathbb{D})$ to be $\mathcal{M}_{H^\infty(\mathbb{D})} \setminus \overline{\mathbb{D}}$.

In 1941, Kakutani asked if there was a Corona in the maximal ideal space $\mathcal{M}_{H^\infty(\mathbb{D})}$ of $H^\infty(\mathbb{D})$, i.e. whether or not the disc $\mathbb{D}$ was dense in $\mathcal{M}_{H^\infty(\mathbb{D})}$?

Where Did the Name Come From?

The Corona Problem for $H^\infty(\mathbb{D})$

Using basic functional analysis, Kakutani's question can be phrased as the following question about analytic functions on the unit disc:

The open disc $\mathbb{D}$ is dense in $\mathcal{M}_{H^\infty(\mathbb{D})}$ (namely the algebra $H^\infty(\mathbb{D})$ has *no* Corona) if and only if the following condition holds:

If $f_1, \ldots, f_N \in H^\infty(\mathbb{D})$ and if

$$\max_{1\leq j\leq N} |f_j(z)| \geq \delta > 0 \quad \forall z \in \mathbb{D}$$

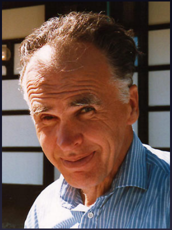
then there exists $g_1, \ldots, g_N \in H^\infty(\mathbb{D})$ such that

$$1 = \sum_{j=1}^{N} f_j(z) g_j(z).$$

Where Did the Name Come From?

The Corona Problem for $H^\infty(\mathbb{D})$

Kakutani's question was settled in 1962 by Carleson: $\overline{\mathbb{D}} = \mathcal{M}_{H^\infty(\mathbb{D})}$.



Lennart Carleson

Theorem (Carleson's Corona Theorem)

Let $\{f_j\}_{j=1}^N \in H^\infty(\mathbb{D})$ *satisfy*

$$0 < \delta \leq \sum_{j=1}^N |f_j(z)|^2 \leq 1, \quad \forall z \in \mathbb{D}.$$

Then there are functions $\{g_j\}_{j=1}^N$ *in* $H^\infty(\mathbb{D})$ *with*

$$\sum_{j=1}^N f_j(z)\, g_j(z) = 1 \; \forall z \in \mathbb{D} \text{ and } \|g_j\|_{H^\infty(\mathbb{D})} \leq C_{\delta,N}.$$

Extensions of the Corona Problem

The point of departure for many generalizations of Carleson's Corona Theorem is the following:

Observation

$H^\infty(\mathbb{D})$ is the (pointwise) multiplier algebra of the classical Hardy space $H^2(\mathbb{D})$ on the unit disc.

Namely, let $M_{H^2}(\mathbb{D})$ denote the class of functions φ such that

$$\|\varphi f\|_{H^2(\mathbb{D})} \leq C\|f\|_{H^2(\mathbb{D})}, \quad \forall f \in H^2(\mathbb{D}). \tag{$\dagger$}$$

with $\|\varphi\|_{M_{H^2}(\mathbb{D})} = \inf\{C : (\dagger) \text{ holds}\}$. Then $\varphi \in H^\infty(\mathbb{D})$ if and only if $\varphi \in M_{H^2}(\mathbb{D})$ and,

$$\|\varphi\|_{M_{H^2}(\mathbb{D})} = \|\varphi\|_{H^\infty(\mathbb{D})}.$$

Besov-Sobolev Spaces

The space $B_2^{\sigma}(\mathbb{B}_n)$ is the collection of holomorphic functions f on the unit ball $\mathbb{B}_n$ such that

$$\left\{ \sum_{k=0}^{m-1} \left| f^{(k)}(0) \right|^2 + \int_{\mathbb{B}_n} \left| \left(1 - |z|^2\right)^{m+\sigma} f^{(m)}(z) \right|^2 d\lambda_n(z) \right\}^{\frac{1}{2}} < \infty,$$

where $d\lambda_n(z) = \left(1 - |z|^2\right)^{-n-1} dV(z)$ is the invariant measure on $\mathbb{B}_n$ and $m + \sigma > \frac{n}{2}$. These spaces can also be defined for $1 < p < \infty$ with appropriate modifications.

Various choices of σ give important examples of classical function spaces:

- $\sigma = 0$: Dirichlet Space;
- $\sigma = \frac{1}{2}$: Drury-Arveson Hardy Space;
- $\sigma = \frac{n}{2}$: Classical Hardy Space;
- $\sigma > \frac{n}{2}$: Bergman Spaces.

Besov-Sobolev Spaces

The spaces $B_2^\sigma(\mathbb{B}_n)$ are examples of reproducing kernel Hilbert spaces. Namely, for each point $\lambda \in \mathbb{B}_n$ there exists a function $k_\lambda \in B_2^\sigma(\mathbb{B}_n)$ such that

$$f(\lambda) = \langle f, k_\lambda \rangle_{B_2^\sigma}.$$

It is an easy computation to show that the kernel function k_λ is given by:

$$k_\lambda^\sigma(z) = \frac{1}{\left(1 - \overline{\lambda} z\right)^{2\sigma}}$$

- $\sigma = 0$: Dirichlet Space; $k_\lambda^0(z) = 1 + \log \frac{1}{1-\overline{\lambda}z}$
- $\sigma = \frac{1}{2}$: Drury-Arveson Hardy Space; $k_\lambda^{\frac{1}{2}}(z) = \frac{1}{1-\overline{\lambda}z}$
- $\sigma = \frac{n}{2}$: Classical Hardy Space; $k_\lambda^{\frac{n}{2}}(z) = \frac{1}{(1-\overline{\lambda}z)^n}$
- $\sigma = \frac{n+1}{2}$: Bergman Space; $k_\lambda^{\frac{n+1}{2}}(z) = \frac{1}{(1-\overline{\lambda}z)^{n+1}}$

Multiplier Algebras of Besov-Sobolev Spaces $M_{B_2^\sigma}(\mathbb{B}_n)$

We are interested in the multiplier algebras, $M_{B_2^\sigma}(\mathbb{B}_n)$, for $B_2^\sigma(\mathbb{B}_n)$. A function φ belongs to $M_{B_2^\sigma}(\mathbb{B}_n)$ if

$$\begin{aligned} \|\varphi f\|_{B_2^\sigma(\mathbb{B}_n)} &\leq C\|f\|_{B_2^\sigma(\mathbb{B}_n)} \quad \forall f\in B_2^\sigma(\mathbb{B}_n) \\ \|\varphi\|_{M_{B_2^\sigma}(\mathbb{B}_n)} &= \inf\{C : \text{above inequality holds}\}. \end{aligned}$$

It is well known that $M_{B_2^\sigma}(\mathbb{B}_n)\subsetneq H^\infty(\mathbb{B}_n)$ when $0\leq\sigma<\frac{n}{2}$.
Let $\mathcal{X}_2^\sigma(\mathbb{B}_n)$ be the functions φ such that for all $f\in B_2^\sigma(\mathbb{B}_n)$:

$$\int_{\mathbb{B}_n}|f(z)|^2\left|\left(1-|z|^2\right)^{m+\sigma}\varphi^{(m)}(z)\right|^2 d\lambda_n(z)\leq C\|f\|_{B_2^\sigma(\mathbb{B}_n)}^2, \qquad (\ddagger)$$

with $\|\varphi\|_{\mathcal{X}_2^\sigma(\mathbb{B}_n)}=\inf\{C : (\ddagger) \text{ holds}\}$. It is easy to see:

$$\begin{aligned} M_{B_2^\sigma}(\mathbb{B}_n) &= H^\infty(\mathbb{B}_n)\cap\mathcal{X}_2^\sigma(\mathbb{B}_n) \\ \|\varphi\|_{M_{B_2^\sigma}(\mathbb{B}_n)} &\approx \|\varphi\|_{H^\infty(\mathbb{B}_n)}+\|\varphi\|_{\mathcal{X}_2^\sigma(\mathbb{B}_n)}. \end{aligned}$$

The Corona Problem for $M_{B_2^\sigma}(\mathbb{B}_n)$

Question (Corona Problem for Multiplier Algebras $M_{B_2^\sigma}(\mathbb{B}_n)$)

Given $f_1, \ldots, f_N \in M_{B_2^\sigma}(\mathbb{B}_n)$ *satisfying*

$$0 < \delta \leq \sum_{j=1}^{N} |f_j(z)|^2 \leq 1 \quad \forall z \in \mathbb{B}_n.$$

Are there functions $g_1, \ldots, g_N \in M_{B_2^\sigma}(\mathbb{B}_n)$ *and a constant* $C_{n,\sigma,N,\delta}$ *such that:*

$$\begin{aligned} \sum_{j=1}^{N} \|g_j\|_{M_{B_2^\sigma}(\mathbb{B}_n)} &\leq C_{n,\sigma,N,\delta} \\ \sum_{j=1}^{N} g_j(z) f_j(z) &= 1 \quad \forall z \in \mathbb{B}_n? \end{aligned}$$

The Baby Corona Problem for $B_2^\sigma(\mathbb{B}_n)$

Question (Baby Corona Problem for $B_2^\sigma(\mathbb{B}_n)$)

Given $f_1, \ldots, f_N \in M_{B_2^\sigma}(\mathbb{B}_n)$ *satisfying*

$$0 < \delta \leq \sum_{j=1}^{N} \left|f_j(z)\right|^2 \leq 1 \quad \forall z \in \mathbb{B}_n$$

and $h \in B_2^\sigma(\mathbb{B}_n)$. *Does there exist a constant* $C_{n,\sigma,N,\delta}$ *and functions* $l_1, \ldots, l_N \in B_2^\sigma(\mathbb{B}_n)$ *satisfying*

$$\begin{aligned} \sum_{j=1}^{N} \left\|l_j\right\|^2_{B_2^\sigma(\mathbb{B}_n)} &\leq C_{n,\sigma,N,\delta} \left\|h\right\|^2_{B_2^\sigma(\mathbb{B}_n)}, \\ \sum_{j=1}^{N} l_j(z) f_j(z) &= h(z) \quad \forall z \in \mathbb{B}_n? \end{aligned}$$

Corona implies Baby Corona

Indeed, if the Corona Problem is true and we take $h \in B_2^\sigma(\mathbb{B}_n)$, we can see that the Baby Corona Problem follows. Suppose $f_1, \ldots, f_N \in M_{B_2^\sigma}(\mathbb{B}_n)$, and there exists $g_1, \ldots, g_N \in M_{B_2^\sigma}(\mathbb{B}_n)$ such that

$$\sum_{j=1}^N \|g_j\|_{M_{B_2^\sigma}(\mathbb{B}_n)} \leq C_{n,\sigma,N,\delta} \qquad \sum_{j=1}^N g_j(z) f_j(z) = 1 \quad \forall z \in \mathbb{B}_n.$$

Multiplying the second equation by h, we find

$$h(z) = \sum_{j=1}^N g_j(z) f_j(z) h(z) = \sum_{j=1}^N l_j(z) f_j(z) \quad \forall z \in \mathbb{B}_n.$$

Since $g_1, \ldots, g_N \in M_{B_2^\sigma}(\mathbb{B}_n)$ we then have that $l_j := g_j h \in B_2^\sigma(\mathbb{B}_n)$ with $\|l_j\|_{B_2^\sigma(\mathbb{B}_n)} \leq \|g_j\|_{M_{B_2^\sigma}(\mathbb{B}_n)} \|h\|_{B_2^\sigma(\mathbb{B}_n)}$, so the claimed estimates follow as well.

Baby Corona implies Corona?

Toeplitz Corona Theorem

Theorem (Toeplitz Corona Theorem, (Agler and McCarthy))

Let $\mathcal{H}$ be a Hilbert function space in an open set Ω in $\mathbb{C}^n$ with an irreducible complete Nevanlinna-Pick kernel. Let $\epsilon > 0$ and let $f_1, \ldots, f_N \in M_{\mathcal{H}}$. Then the following are equivalent:

(i) *There exists $g_1, \ldots, g_N \in M_{\mathcal{H}}$ such that $\sum_{j=1}^{N} f_j g_j = 1$ and $\sum_{j=1}^{N} \|g_j\|_{M_{\mathcal{H}}} \leq \frac{1}{\epsilon}$;*

(ii) *For any $h \in \mathcal{H}$, there exists $l_1, \ldots, l_N \in \mathcal{H}$ such that $h = \sum_{j=1}^{N} l_j f_j$ and $\sum_{j=1}^{N} \|l_j\|_{\mathcal{H}}^2 \leq \frac{1}{\epsilon^2} \|h\|_{\mathcal{H}}^2$.*

Moral: If the Hilbert space has a reproducing kernel with enough structure, then the Corona Problem and the Baby Corona Problem are the same question.

Baby Corona Theorem for $B_p^\sigma(\mathbb{B}_n)$

Theorem (Ş. Costea, E. Sawyer, BDW; Analysis & PDE **4** (2011))

Let $0 \leq \sigma$ *and* $1 < p < \infty$. *Given* $f_1, \ldots, f_N \in M_{B_p^\sigma}(\mathbb{B}_n)$ *satisfying*

$$0 < \delta \leq \sum_{j=1}^{N} |f_j(z)|^2 \leq 1, \qquad z \in \mathbb{B}_n,$$

and $h \in B_p^\sigma(\mathbb{B}_n)$. *There are functions* $l_1, \ldots, l_N \in B_p^\sigma(\mathbb{B}_n)$ *and a constant* $C_{n,\sigma,N,p,\delta}$ *such that*

$$\begin{aligned} \sum_{j=1}^{N} \|l_j\|_{B_p^\sigma(\mathbb{B}_n)}^p &\leq C_{n,\sigma,N,p,\delta} \|h\|_{B_p^\sigma(\mathbb{B}_n)}^p, \\ \sum_{j=1}^{N} l_j(z) f_j(z) &= h(z) \quad \forall z \in \mathbb{B}_n. \end{aligned}$$

The Corona Theorem for $M_{B_2^\sigma}(\mathbb{B}_n)$

Theorem (Ş. Costea, E. Sawyer, BDW; Analysis & PDE 4 (2011))

Let $0 \leq \sigma \leq \frac{1}{2}$ *and* $p = 2$. *Given* $f_1, \ldots, f_N \in M_{B_2^\sigma}(\mathbb{B}_n)$ *satisfying*

$$0 < \delta \leq \sum_{j=1}^{N} |f_j(z)|^2 \leq 1 \quad \forall z \in \mathbb{B}_n,$$

there are functions $g_1, \ldots, g_N \in M_{B_2^\sigma}(\mathbb{B}_n)$ *and a constant* $C_{n,\sigma,N,\delta}$ *such that*

$$\begin{aligned} \sum_{j=1}^{N} \|g_j\|_{M_{B_2^\sigma}(\mathbb{B}_n)} &\leq C_{n,\sigma,N,\delta} \\ \sum_{j=1}^{N} g_j(z) f_j(z) &= 1, \qquad z \in \mathbb{B}_n. \end{aligned}$$

The Corona Theorem for $M_{B_2^\sigma}(\mathbb{B}_n)$

The proof of the Corona Theorem follows very easily from the Baby Corona Theorem:

- When $0 \leq \sigma \leq \frac{1}{2}$ the spaces $B_2^\sigma(\mathbb{B}_n)$ are reproducing kernel Hilbert spaces with a complete Nevanlinna-Pick kernel;
- By the Toeplitz Corona Theorem, we then have that the Baby Corona Problem is equivalent to the full Corona Problem. The result then follows.

An additional corollary of the above result is the following:

Corollary

For $0 \leq \sigma \leq \frac{1}{2}$, the unit ball $\mathbb{B}_n$ is dense in the maximal ideal space of $M_{B_2^\sigma}(\mathbb{B}_n)$.

This is because the density of the unit ball $\mathbb{B}_n$ in the maximal ideal space of $M_{B_2^\sigma}(\mathbb{B}_n)$ is equivalent to the Corona Theorem above.

Sketch of Proof of the Baby Corona Theorem

Given $h \in M_{B_p^\sigma}(\mathbb{B}_n)$ and $f_1, \ldots, f_N \in M_{B_p^\sigma}(\mathbb{B}_n)$ satisfying

$$0 < \delta \leq \sum_{j=1}^{N} |f_j(z)|^2 \leq 1, \qquad z \in \mathbb{B}_n.$$

Set $\varphi_j(z) = \frac{\overline{f_j(z)}}{\sum_j |f_j(z)|^2} h(z)$. We have that $\sum_{j=1}^N f_j(z)\varphi_j(z) = h(z)$.

In order to have an analytic solution we will need to solve a sequence of $\overline{\partial}$-equations: The Koszul Complex.

This gives an algorithmic way of solving the $\overline{\partial}$-equations for each $(0, q)$ with $1 \leq q \leq n$ after starting with a $(0, n)$ form.

The Koszul Complex gives us $l_j = \varphi_j - \xi_j$.

Algebraic properties of the Koszul complex give that $\sum_{j=1}^N f_j l_j = h$.

Sketch of Proof of the Corona Theorem for Multiplier Algebras

Hard work then lets you conclude that the solutions obtained by the Koszul complex have the desired estimates.

Key Ideas in the Proof:

- Exact structure of the kernel of the solution operator that takes $(0, q)$ forms to $(0, q-1)$ forms:

$$\frac{(1 - w\overline{z})^{n-q}\left(1 - |w|^2\right)^{q-1}}{\triangle(w, z)^n}(\overline{w}_j - \overline{z}_j) \quad \forall\, 1 \leq q \leq n.$$

  Here $\triangle(w, z) = |1 - w\overline{z}|^2 - \left(1 - |w|^2\right)\left(1 - |z|^2\right)$.

- The solution operators to the $\overline{\partial}$-problem take the Besov-Sobolev spaces $B_p^\sigma(\mathbb{B}_n)$ to themselves.

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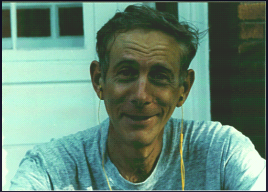

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Bilinear Forms on the Hardy Space

- The Hardy space $H^2(\mathbb{D})$ is the collection of all analytic functions on the disc such that

$$\|f\|_{H^2}^2 := \sup_{0<r<1} \int_{\mathbb{T}} |f(r\xi)|^2 \, dm(\xi) < \infty.$$

- The Hankel Operator H_b maps $H^2(\mathbb{D})$ to $H^2(\mathbb{D})^\perp$ and is given by

$$H_b := (I - \mathbb{P}_{H^2}) \, M_b.$$

- To study the boundedness of this operator, we can study only the corresponding bilinear Hankel form $T_b : H^2(\mathbb{D}) \times H^2(\mathbb{D}) \to \mathbb{C}$,

$$T_b(f, g) := \langle fg, b \rangle_{H^2}.$$

Bilinear Forms on the Hardy Space

- The bilinear form T_b is bounded if and only if b belongs to $BMOA(\mathbb{D})$.
- We can connect this to Carleson measures for the space $H^2(\mathbb{D})$.

Lemma

A function $b \in BMOA(\mathbb{D})$ if and only if $b \in H^2(\mathbb{D})$ and

$$|b'(z)|^2(1-|z|^2)dA(z)$$

is a Carleson measure for $H^2(\mathbb{D})$.

Theorem

The bilinear form $T_b : H^2(\mathbb{D}) \times H^2(\mathbb{D}) \to \mathbb{C}$ is bounded if and only if

$$|b'(z)|^2(1-|z|^2)dA(z)$$

is a Carleson measure for $H^2(\mathbb{D})$.

The Dirichlet Space $\mathcal{D}$

Definition (Dirichlet Space)

An analytic function is an element of the Dirichlet space $\mathcal{D}$ if and only if

$$\|f\|_{\mathcal{D}}^2 := |f(0)|^2 + \int_{\mathbb{D}} |f'(z)|^2 \, dA(z) < \infty.$$

This is this the space $B_2^0(\mathbb{D})$ introduced before.

Definition (Carleson Measures for the Dirichlet Space)

A measure μ on $\mathbb{D}$ is a $\mathcal{D}$-Carleson measure if and only if

$$\int_{\mathbb{D}} |f(z)|^2 \, d\mu(z) \leq \|\mu\|_{\mathcal{D}-\text{Carleson}}^2 \|f\|_{\mathcal{D}}^2$$

for all $f \in \mathcal{D}$.

This is related to the space $\mathcal{X}_2^0(\mathbb{D})$ introduced before.

Carleson Measures for $\mathcal{D}$

Logarithmic Capacity on the Disc

For an interval $I \subset \mathbb{T}$, let $T(I)$ be the Carleson tent over the interval I,

$$T(I) := \left\{ z \in \mathbb{D} : 1 - |I| \leq |z| \leq 1, \frac{z}{|z|} \in I \right\}.$$

This definition obviously extends to general compact sets $E \subset \mathbb{T}$.
Given a compact subset $E \subset \mathbb{T}$, the capacity of the set E is defined by

$$\operatorname{cap}(E) := \inf \left\{ \|f\|_{\mathcal{D}}^2 : \operatorname{Re} f \geq 1 \text{ on } T(E) \right\}.$$

It is easy to see that for an interval $I \subset \mathbb{T}$ we have

$$\operatorname{cap}(I) \approx \left(\log \left(\frac{2\pi}{|I|} \right) \right)^{-1}.$$

Carleson Measures for $\mathcal{D}$: Geometric Characterization

There is an obvious necessary condition a $\mathcal{D}$-Carleson must satisfy: Suppose that μ is a $\mathcal{D}$-Carleson measure. For $\lambda \in \mathbb{D}$, let

$$k_\lambda(z) := 1 + \log \frac{1}{1 - \overline{\lambda} z}.$$

Then $k_\lambda \in \mathcal{D}$ and $\|k_\lambda\|_{\mathcal{D}}^2 \approx -\log\left(1 - |\lambda|^2\right)$. Let $\tilde{k}_\lambda$ denote the (approximately) normalized version of k_λ, i.e. $\|\tilde{k}_\lambda\|_{\mathcal{D}} \approx 1$.

For each interval $I \subset \mathbb{T}$ there exists a unique $\lambda \in \mathbb{D}$ with $1 - |\lambda|^2 = |I|$.

Standard estimates show:

$$\frac{\mu\left(T(I)\right)}{\text{cap}(I)} \lesssim \int_{\mathbb{D}} |\tilde{k}_\lambda(z)|^2 \, d\mu(z) \leq \|\mu\|_{\mathcal{D}-\text{Carleson}}^2 \|\tilde{k}_\lambda\|_{\mathcal{D}}^2 \approx \|\mu\|_{\mathcal{D}-\text{Carleson}}^2.$$

Unfortunately, this simple condition is not sufficient.

Carleson Measures for $\mathcal{D}$: Geometric Characterization

Theorem (Stegenga (1980))

A measure μ is a Carleson measure for $\mathcal{D}$ if and only if

$$\mu\left(\cup_{j=1}^N T(I_j)\right) \leq S(\mu)\,\mathrm{cap}\left(\cup_{j=1}^N I_j\right),$$

for all finite unions of disjoint arcs on the boundary $\mathbb{T}$.

- This is a geometric characterization of the Carleson measures.
- But, it is difficult to check:
  - Computing capacity is hard.
  - One has to check every possible collection of disjoint intervals in $\mathbb{T}$.
- One has an equivalence between the quantities $\|\mu\|^2_{\mathcal{D}-\mathrm{Carleson}}$ and $S(\mu)$,

$$\|\mu\|^2_{\mathcal{D}-\mathrm{Carleson}} \approx S(\mu).$$

Hankel Operators on the Dirichlet Space

In an analogous manner, one defines the (small) Hankel operator $h_b : \mathcal{D} \to \overline{\mathcal{D}}$ by

$$h_b := \overline{\mathbb{P}_{\mathcal{D}} M_b} = \int_{\mathbb{D}} \overline{b'(z)} f'(z) g(z) \, dA(z)$$

Definition

Suppose that b is analytic on $\mathbb{D}$. It belongs to $\mathcal{X}(\mathbb{D})$ if and only if

$$\int_{\mathbb{D}} |f(z)|^2 |b'(z)|^2 \, dA(z) \leq C^2 \|f\|_{\mathcal{D}}^2, \quad \forall f \in \mathcal{D}. \tag{$\dagger$}$$

Moreover, we norm the space by

$$\|b\|_{\mathcal{X}} := \inf\{C : (\dagger) \text{ holds}\} + |b(0)|$$

Namely, $d\mu_b(z) := |b'(z)|^2 \, dA(z)$ is a $\mathcal{D}$-Carleson measure and

$$\|b\|_{\mathcal{X}} = \|\mu_b\|_{\mathcal{D}-\text{Carleson}} + |b(0)|.$$

Hankel Operators on the Dirichlet Space

Theorem (Rochberg, Wu (1993))

Suppose that b is analytic on $\mathbb{D}$. Then h_b is bounded if and only if $b \in \mathcal{X}(\mathbb{D})$. Moreover,

$$\|h_b\|_{\mathcal{D}\to\overline{\mathcal{D}}} \approx \|\mu_b\|_{\mathcal{D}-\text{Carleson}}.$$

One can also look at the corresponding problem for the bilinear form

$$T_b : \mathcal{D} \times \mathcal{D} \to \mathbb{C}.$$

But, one can easily observe that the operator h_b does not induce the bilinear form T_b.

Conjecture

The bilinear form $T_b : \mathcal{D} \times \mathcal{D} \to \mathbb{C}$ is bounded if and only if $b \in \mathcal{X}(\mathbb{D})$.

Bounded Bilinear Forms on the Dirichlet Space

Theorem (N. Arcozzi, R. Rochberg, E. Sawyer, BDW; Analysis & PDE **3** (2010))

Let $T_b : \mathcal{D} \times \mathcal{D} \to \mathbb{C}$ *be the bilinear form defined by*

$$\begin{aligned} T_b(f,g) &:= \langle fg, b\rangle_{\mathcal{D}} \\ &= f(0)g(0)\overline{b(0)} + \int_{\mathbb{D}} \overline{b'(z)}\left(f'(z)g(z) + f(z)g'(z)\right) dA(z). \end{aligned}$$

Let $d\mu_b(z) := |b'(z)|^2 dA(z)$. *Then* T_b *is a bounded bilinear form on* $\mathcal{D} \times \mathcal{D}$ *if and only if* $b \in \mathcal{X}(\mathbb{D})$ *with*

$$\|b\|_{\mathcal{X}} := \|\mu_b\|_{\mathcal{D}-\text{Carleson}} + |b(0)| \approx \|T_b\|_{\mathcal{D}\times\mathcal{D}\to\mathbb{C}}.$$

This Theorem demonstrates that the corresponding picture for the Hardy space $H^2(\mathbb{D})$ carries over to $\mathcal{D}$.

Carleson Measure $\implies$ Bounded Bilinear Form

Suppose that μ_b is a $\mathcal{D}$-Carleson measure. For $f, g \in \mathrm{Pol}\,(\mathbb{D})$ we have

$$T_b(f,g) := f(0)g(0)\overline{b(0)} + \textstyle\int_{\mathbb{D}} \overline{b'(z)}\,(f'(z)g(z) + f(z)g'(z))\,dA(z).$$

$$\begin{aligned}
|T_b(f,g)| &\leq |f(0)g(0)\overline{b(0)}| + \int_{\mathbb{D}} |f'(z)g(z)\overline{b'(z)}|dA(z) \\
&\quad + \int_{\mathbb{D}} |f(z)g'(z)\overline{b'(z)}|dA(z) \\
&\leq |f(0)g(0)\overline{b(0)}| + \|f\|_{\mathcal{D}} \left(\int_{\mathbb{D}} |g(z)|^2 d\mu_b(z)\right)^{\frac{1}{2}} \\
&\quad + \|g\|_{\mathcal{D}} \left(\int_{\mathbb{D}} |f(z)|^2 d\mu_b(z)\right)^{\frac{1}{2}} \\
&\leq 2\,(|b(0)| + \|\mu_b\|_{\mathcal{D}-\mathrm{Carleson}})\,\|f\|_{\mathcal{D}}\|g\|_{\mathcal{D}} \\
&= 2\|b\|_{\mathcal{X}}\|f\|_{\mathcal{D}}\|g\|_{\mathcal{D}}.
\end{aligned}$$

So, T_b has a bounded extension from $\mathcal{D} \times \mathcal{D} \to \mathbb{C}$ with

$$\|T_b\|_{\mathcal{D}\times\mathcal{D}\to\mathbb{C}} \lesssim \|b\|_{\mathcal{X}}.$$

Bounded Bilinear Form $\Longrightarrow$ Carleson Measure

Choose an (almost) extremal collection of intervals $\{I_j\}_j \subset \mathbb{T}$ so that we have

$$S(\mu_b) := \sup \frac{\mu_b\left(\cup_{j=1}^N T(I_j)\right)}{\operatorname{cap}(\cup_{j=1}^N I_j)} = \frac{\mu_b\left(\cup_{j=1}^N T(I_j)\right)}{\operatorname{cap}(\cup_{j=1}^N I_j)}.$$

We will use this collection of intervals to construct functions f and g to test in the bilinear for T_b. One then proves an estimate of the form:

$$\frac{\mu_b\left(\cup_{j=1}^N T(I_j)\right)}{\operatorname{cap}(\cup_{j=1}^N I_j)} \lesssim \| T_b\|^2_{\mathcal{D}\times\mathcal{D}\to\mathbb{C}}.$$

The function g will be constructed using an approximate extremal function from the collection of intervals that achieves the supremum and will be approximately equal to the indicator function on $\cup_{j=1}^N T(I_j)$. The function f will be, approximately, b' on the set $\cup_{j=1}^N T(I_j)$.

Bounded Bilinear Form $\Longrightarrow$ Carleson Measure

Using the extremal intervals we selected, we can form a holomorphic function φ that is basically the indicator of $\cup_j T(I_j)$.

Lemma

There exists a holomorphic function φ such that

$$\begin{cases} |\varphi(z)-\varphi(w_k^\alpha)| & \lesssim & \mathrm{cap}(\cup_{j=1}^N I_j), & z\in T(I_k^\alpha) \\ \mathrm{Re}\,\varphi(w_k^\alpha) & \geq & c>0, & 1\leq k\leq M_\alpha \\ |\varphi(w_k^\alpha)| & \leq & C, & 1\leq k\leq M_\alpha \\ |\varphi(z)| & \lesssim & \mathrm{cap}(\cup_{j=1}^N I_j), & z\notin \cup_{j=1}^N T\left(I_j^\gamma\right). \end{cases}$$

Moreover,

$$\|\varphi\|_{\mathcal{D}}^2\lesssim \mathrm{cap}\left(\cup_{j=1}^N I_j\right).$$

Bounded Bilinear Form $\Longrightarrow$ Carleson Measure

We will use $g = \varphi^2$ and

$$f(z) := \int_{\cup_{j=1}^{N} T(I_j)} \frac{b'(\zeta)}{\left(1 - \overline{\zeta} z\right)} \frac{dA(\zeta)}{\overline{\zeta}}.$$

Using the reproducing kernel property we find that

$$\begin{aligned} f'(z) &= \int_{\cup_{j=1}^{N} T(I_j)} \frac{b'(\zeta)}{\left(1 - \overline{\zeta} z\right)^2} dA(\zeta) \\ &= b'(z) - \int_{\mathbb{D} \setminus \cup_{j=1}^{N} T(I_j)} \frac{b'(\zeta)}{\left(1 - \overline{\zeta} z\right)^2} dA(\zeta) \\ &=: b'(z) + \Lambda b'(z). \end{aligned}$$

This function f is approximately b' on the set $\cup_{j=1}^{N} T(I_j)$.

Bounded Bilinear Form $\Longrightarrow$ Carleson Measure

If we substitute these into the bilinear form T_b we find that:

$$\begin{aligned} T_b(f,g) &= T_b(f,\varphi^2) = T_b(f\varphi,\varphi) \\ &= \int_{\mathbb{D}} \left\{f'(z)\varphi(z) + 2f(z)\varphi'(z)\right\} \varphi(z)\overline{b'(z)}\,dA(z) \\ &\quad +f(0)\varphi(0)^2\overline{b(0)} \\ &= f(0)\varphi(0)^2\overline{b(0)} + \int_{\mathbb{D}} |b'(z)|^2\varphi(z)^2\,dA(z) \\ &\quad +2\int_{\mathbb{D}} \varphi(z)\varphi'(z)f(z)\overline{b'(z)}\,dA(z) + \int_{\mathbb{D}} \Lambda b'(z)\overline{b'(z)}\varphi(z)^2\,dA(z) \\ &:= (1)+(2)+(3)+(4). \end{aligned}$$

First by selection of the function f and g one easily shows that

$$|T_b(f,g)| \lesssim \|T_b\|^2_{\mathcal{D}\times\mathcal{D}\to\mathbb{C}}\,\mathrm{cap}\left(\cup_{j=1}^N T(I_j)\right).$$

Bounded Bilinear Form $\implies$ Carleson Measure

- Term (1) is trivial.
- Term (2) yields (using properties of φ and a geometric property)

$$\begin{aligned}
(2) &= \int_{\mathbb{D}} |b'(z)|^2 \varphi(z)^2 \, dA(z) \\
&= \left\{ \int_{\cup_{j=1}^N T(I_j)} + \int_{\cup_{j=1}^N T(I_j^\beta) \setminus \cup_{j=1}^N T(I_j)} \right\} |b'(z)|^2 \varphi(z)^2 \, dA \\
&\quad + \int_{\mathbb{D} \setminus \cup_{j=1}^N T(I_j^\beta)} |b'(z)|^2 \varphi(z)^2 \, dA \\
&=: (2_A) + (2_B) + (2_C).
\end{aligned}$$

- The main term (2_A) satisfies

$$\begin{aligned}
(2_A) &= \mu_b\left(\cup_{j=1}^N T(I_j)\right) + \int_{\cup_{j=1}^N T(I_j)} |b'(z)|^2 \left(\varphi(z)^2 - 1\right) dA(z) \\
&= \mu_b\left(\cup_{j=1}^N T(I_j)\right) + O\left(\|T_b\|^2 \operatorname{cap}\left(\cup_{j=1}^N T(I_j)\right)\right).
\end{aligned}$$

Bounded Bilinear Form $\implies$ Carleson Measure

- Terms (2_B) and (2_C) are error terms controlled by properties of φ.
- Terms (3) and (4) are also error terms.
- Using properties of φ, geometric estimates, and Schur's Lemma, we can show these errors are controlled by estimates of the form

$$\epsilon\mu_b\left(\cup_{j=1}^N T(I_j)\right) + C(\epsilon)\| T_b\|^2_{\mathcal{D}\times\mathcal{D}\to\mathbb{C}} \operatorname{cap}\left(\cup_{j=1}^N I_j\right)$$

  where $\epsilon > 0$ is a small number to be chosen later.
- Thus, we have

$$\mu_b\left(\cup_{j=1}^N T(I_j)\right) \lesssim \epsilon\mu_b\left(\cup_{j=1}^N T(I_j)\right) + C(\epsilon)\| T_b\|^2_{\mathcal{D}\times\mathcal{D}\to\mathbb{C}} \operatorname{cap}\left(\cup_{j=1}^N I_j\right).$$

- Choosing $\epsilon > 0$ small enough gives

$$\mu_b\left(\cup_{j=1}^N T(I_j)\right) \lesssim \| T_b\|^2_{\mathcal{D}\times\mathcal{D}\to\mathbb{C}} \operatorname{cap}\left(\cup_{j=1}^N I_j\right).$$

Thank You!