Carleson Measures for Besov-Sobolev Spaces and Non-Homogeneous Harmonic Analysis

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Talk Outline

- Motivation of the Problem
  - Besov-Sobolev Spaces of analytic functions on $\mathbb{B}_n$
  - Carleson Measures for Besov-Sobolev Spaces
  - Connections to Non-Homogeneous Harmonic Analysis
- Main Results and Sketch of Proof
  - T(1)-Theorem for Bergman-type operators
  - Characterization of Carleson measures for Besov-Sobolev Spaces
- Further Results

Besov-Sobolev Spaces

- The space $B_2^\sigma(\mathbb{B}_n)$ is the collection of holomorphic functions f on the unit ball $\mathbb{B}_n := \{z \in \mathbb{C}^n : |z| < 1\}$ such that

$$\left\{\sum_{k=0}^{m-1}\left|f^{(k)}(0)\right|^2 + \int_{\mathbb{B}_n}\left|\left(1-|z|^2\right)^{m+\sigma} f^{(m)}(z)\right|^2 d\lambda_n(z)\right\}^{\frac{1}{2}} < \infty,$$

  where $d\lambda_n(z) = \left(1-|z|^2\right)^{-n-1} dV(z)$ is the invariant measure on $\mathbb{B}_n$ and $m+\sigma > \frac{n}{2}$.
- Various choices of σ give important examples of classical function spaces:
  - $\sigma = 0$: Corresponds to the Dirichlet Space;
  - $\sigma = \frac{1}{2}$: Drury-Arveson Hardy Space;
  - $\sigma = \frac{n}{2}$: Classical Hardy Space;
  - $\sigma > \frac{n}{2}$: Bergman Spaces.

Besov-Sobolev Spaces

- The spaces $B_2^\sigma(\mathbb{B}_n)$ are examples of reproducing kernel Hilbert spaces.
- Namely, for each point $\lambda \in \mathbb{B}_n$ there exists a function $k_\lambda^\sigma \in B_2^\sigma(\mathbb{B}_n)$ such that

$$f(\lambda) = \langle f, k_\lambda^\sigma \rangle_{B_2^\sigma(\mathbb{B}_n)}$$

- A computation shows that the kernel function $k_\lambda^\sigma(z)$ is given by:

$$k_\lambda^\sigma(z) = \frac{1}{\left(1 - \overline{\lambda} z\right)^{2\sigma}}$$

  - $\sigma = \frac{1}{2}$: Drury-Arveson Hardy Space; $k_\lambda^{\frac{1}{2}}(z) = \frac{1}{1-\overline{\lambda}z}$
  - $\sigma = \frac{n}{2}$: Classical Hardy Space; $k_\lambda^{\frac{n}{2}}(z) = \frac{1}{(1-\overline{\lambda}z)^{n}}$
  - $\sigma = \frac{n+1}{2}$: Bergman Space; $k_\lambda^{\frac{n+1}{2}}(z) = \frac{1}{(1-\overline{\lambda}z)^{n+1}}$

Carleson Measures for Besov-Sobolev Spaces

Definition

A non-negative Borel measure μ is a $B_2^\sigma(\mathbb{B}_n)$-Carleson measure if

$$\int_{\mathbb{B}_n} |f(z)|^2 d\mu(z) \leq C(\mu)^2 \|f\|^2_{B_2^\sigma(\mathbb{B}_n)} \quad \forall f \in B_2^\sigma(\mathbb{B}_n).$$

As is well-known Carleson measures play an important role in both the function theory of the space (e.g., characterization of the multipliers for the space and the interpolating sequences) and more generally play a predominant role in harmonic analysis.

The definition is in terms of function theoretic information, and it would be more useful to have a "testable" condition to check.

Carleson Measures for Besov-Sobolev Spaces

Question

Give a 'geometric' characterization of these measures.

Testing on the reproducing kernel k_λ^σ we always have a necessary geometric condition for the measure μ to be Carleson:

$$\mu\left(T(B_r(\xi))\right) \lesssim r^{2\sigma} \quad \forall \xi \in \partial\mathbb{B}_n,\ r > 0$$

Here $T(B_r(\xi))$ is the tent over the ball or radius $r > 0$ in the boundary $\partial\mathbb{B}_n$

When $\sigma \geq \frac{n}{2}$ then this necessary condition is also sufficient.

Carleson Measures for Besov-Sobolev Spaces

When $0 \leq \sigma \leq \frac{1}{2}$ then a geometric characterization of Carleson measures is known.

- If $n = 1$, then the results can be expressed in terms of capacity conditions. More precisely,

$$\mu(T(\Omega)) \lesssim \mathrm{Cap}_\sigma(\Omega) \quad \forall \text{open}\, \Omega \subset \mathbb{T}.$$

  See for example Stegenga, Maz'ya, Verbitsky, Carleson.
- If $n > 1$ then there are two different characterizations of Carleson measures for $B_2^\sigma(\mathbb{B}_n)$:
  - One method via integration operators on trees (dyadic structures on the ball $\mathbb{B}_n$) by Arcozzi, Rochberg and Sawyer.
  - One method via "T(1)" conditions by E. Tchoundja.

Question (Main Problem: Characterization in the Difficult Range)

Characterize the Carleson measures when $\frac{1}{2} < \sigma < \frac{n}{2}$.

Operator Theoretic Characterization of Carleson Measures

The following observations hold in an arbitrary Hilbert space with a reproducing kernel.

- Let $\mathcal{J}$ be a Hilbert space of functions on a domain X with reproducing kernel function j_x, i.e.,

$$f(x) = \langle f, j_x \rangle_{\mathcal{J}} \quad \forall f \in \mathcal{J}.$$

- A measure μ is Carleson exactly if the inclusion map ι from $\mathcal{J}$ to $L^2(X;\mu)$ is bounded, or

$$\int_X |f(x)|^2 d\mu(z) \leq C(\mu) \|f\|_{\mathcal{J}}^2.$$

We can give a characterization of Carleson measures for the space $\mathcal{J}$ in terms of information about the boundedness of a certain linear operator related to the reproducing kernel j_x.

Operator Theoretic Characterization of Carleson Measures

Statement and Proof

Proposition

A measure μ is a $\mathcal{J}$-Carleson measure if and only if the linear map

$$f(z) \to T(f)(z) = \int_X \operatorname{Re} j_x(z) f(x) d\mu(x)$$

is bounded on $L^2(X;\mu)$.

Proof:

The inclusion map ι is bounded from $\mathcal{J}$ to $L^2(X;\mu)$ if and only if the adjoint map ι^* is bounded from $L^2(X;\mu)$ to $\mathcal{J}$, namely,

$$\|\iota^* f\|_{\mathcal{J}}^2 = \langle \iota^* f, \iota^* f\rangle_{\mathcal{J}} \leq C\|f\|_{L^2(X;\mu)}^2, \quad \forall f \in L^2(X;\mu).$$

Operator Theoretic Characterization of Carleson Measures

Proof of Proposition

For an $x \in X$ we have

$$\begin{aligned} \iota^* f(x) = \langle \iota^* f, j_x \rangle_{\mathcal{J}} &= \langle f, \iota j_x \rangle_{L^2(X:\mu)} \\ &= \int_X f(w)\overline{j_x(w)}d\mu(w) \\ &= \int_X f(w) j_w(x) d\mu(w). \end{aligned}$$

Using this computation, we obtain that

$$\begin{aligned} \|\iota^* f\|_{\mathcal{J}}^2 &= \langle \iota^* f, \iota^* f \rangle_{\mathcal{J}} \\ &= \left\langle \int_X j_w f(w) d\mu(w), \int_X j_{w'} f(w') d\mu(w') \right\rangle_{\mathcal{J}} \end{aligned}$$

Operator Theoretic Characterization of Carleson Measures

Proof of Proposition

Those computations then give

$$\begin{aligned}\|\iota^* f\|_{\mathcal{J}}^2 &= \int_X \int_X \langle j_w, j_{w'} \rangle_{\mathcal{J}} f(w) d\mu(w) \overline{f(w')} d\mu(w') \\ &= \int_X \int_X j_w(w') f(w) d\mu(w) \overline{f(w')} d\mu(w').\end{aligned}$$

The continuity of ι^* for general f is equivalent to having it for real f, without loss of generality, we can suppose that f is real. In that case we can continue the estimates with

$$\|\iota^* f\|_{\mathcal{J}}^2 = \int_X \int_X \operatorname{Re} j_w(w') f(w) f(w') d\mu(w) d\mu(w') = \langle Tf, f \rangle_{L^2(X;\mu)}.$$

But the last quantity satisfies the required estimates exactly when the operator T is bounded. $\square$

Connections to Calderón-Zygmund Operators

When we apply this proposition to the spaces $B_2^{\sigma}(\mathbb{B}_n)$ this suggests that we study the operator

$$T_{\mu,2\sigma}(f)(z) = \int_{\mathbb{B}_n} \operatorname{Re}\left(\frac{1}{(1-\overline{w}z)^{2\sigma}}\right) f(w)d\mu(w) : L^2(\mathbb{B}_n;\mu) \to L^2(\mathbb{B}_n;\mu)$$

and find some conditions that will let us determine when it is bounded. Recall the following theorem of David and Journé:

Theorem (David and Journé)

If T is a Calderón-Zygmund operator then $T : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ if and only if $T(1), T^*(1) \in BMO(\mathbb{R}^n)$ and T is weak bounded.

Idea: Try to use T(1) to characterize the boundedness of

$$T_{\mu,2\sigma} : L^2(\mathbb{B}_n;\mu) \to L^2(\mathbb{B}_n;\mu).$$

Calderón–Zygmund Estimates for $T_{\mu,2\sigma}$

If we define

$$\Delta(z,w) := \begin{cases} \left||z|-|w|\right| + \left|1-\frac{z\overline{w}}{|z||w|}\right| & : \quad z,w\in\mathbb{B}_n\setminus\{0\} \\ |z|+|w| & : \quad \text{otherwise.} \end{cases}$$

Then Δ is a pseudo-metric and makes the ball into a space of homogeneous type.
A computation demonstrates that the kernel of $T_{\mu,2\sigma}$ satisfies the following estimates:

$$|K_{2\sigma}(z,w)| \lesssim \frac{1}{\Delta(z,w)^{2\sigma}} \quad \forall z,w\in\mathbb{B}_n;$$

If $\Delta(\zeta,w) < \frac{1}{2}\Delta(z,w)$ then

$$|K_{2\sigma}(\zeta,w) - K_{2\sigma}(z,w)| \lesssim \frac{\Delta(\zeta,w)^{1/2}}{\Delta(z,w)^{2\sigma+1/2}}.$$

Calderón-Zygmund Estimates for $T_{\mu,2\sigma}$

- These estimates on $K_{2\sigma}(z,w)$ say that it is a Calderón-Zygmund kernel of order 2σ with respect to the metric Δ.
  - Unfortunately, we can't apply the standard T(1) technology (adapted to a space of homogeneous type) to study the operators $T_{\mu,2\sigma}$. We would need the estimates of order n instead of 2σ.
- However, the measures we want to study (the Carleson measures for the space) satisfy the growth estimate

$$\mu\left(T(B_r)\right) \lesssim r^{2\sigma}$$

  and this is exactly the issue that will save us!
- This places us in the setting of non-homogeneous harmonic analysis as developed by Nazarov, Treil and Volberg. We have an operator with a Calderón-Zygmund kernel satisfying estimates of order 2σ, a measure μ of order 2σ, and are interested in $L^2(\mathbb{B}_n;\mu) \to L^2(\mathbb{B}_n;\mu)$ bounds.

Euclidean Variant of the Question

Their is a natural extension of these questions/ideas to the Euclidean setting $\mathbb{R}^d$.

More precisely, for $m \leq d$ we are interested in Calderón-Zygmund kernels that satisfy the following estimates:

$$|k(x,y)| \leq \frac{C_{CZ}}{|x-y|^m},$$

and

$$|k(y,x) - k(y,x')| + |k(x,y) - k(x',y)| \leq C_{CZ} \frac{|x-x'|^\tau}{|x-y|^{m+\tau}}$$

provided that $|x-x'| \leq \frac{1}{2}|x-y|$, with some (fixed) $0 < \tau \leq 1$ and $0 < C_{CZ} < \infty$.

Euclidean Variant of the Question

Additionally the kernels will have the following property

$$|k(x,y)| \leq \frac{1}{\max(d(x)^m, d(y)^m)},$$

where $d(x) := \text{dist}(x, \mathbb{R}^d \setminus H)$ and H being an open set in $\mathbb{R}^d$.
Key examples: Let $H = \mathbb{B}_d$, the unit ball in $\mathbb{R}^d$ and

$$k(x,y) = \frac{1}{(1 - x \cdot y)^m}.$$

We will say that k is a Calderón-Zygmund kernel on a closed $X \subset \mathbb{R}^d$ if $k(x,y)$ is defined only on $X \times X$ and the previous properties of k are satisfied whenever $x, x', y \in X$.
Once the kernel has been defined, then we say that a $L^2(\mathbb{R}^d; \mu)$ bounded operator is a Calderón-Zygmund operator with kernel k if,

$$T_{\mu,m} f(x) = \int_{\mathbb{R}^d} k(x,y) f(y) d\mu(y) \quad \forall x \notin \text{supp} f .$$

Main Results

T(1)-Theorem for Bergman-Type Operators

Theorem (T(1)-Theorem for Bergman-Type Operators, Volberg and Wick (2009))

Let $k(x,y)$ be a Calderón-Zygmund kernel of order m on $X \subset \mathbb{R}^d$, $m \leq d$ with Calderón-Zygmund constants C_{CZ} and τ. Let μ be a probability measure with compact support in X and all balls such that $\mu(B_r(x)) > r^m$ lie in an open set H. Let also

$$|k(x,y)| \leq \frac{1}{\max(d(x)^m, d(y)^m)},$$

where $d(x) := \mathrm{dist}(x, \mathbb{R}^d \setminus H)$. Finally, suppose also that:

$$\| T_{\mu,m}\chi_Q\|^2_{L^2(\mathbb{R}^d;\mu)} \leq A\,\mu(Q)\,,\ \| T^*_{\mu,m}\chi_Q\|^2_{L^2(\mathbb{R}^d;\mu)} \leq A\,\mu(Q)\,.$$

Then $\| T_{\mu,m}\|_{L^2(\mathbb{R}^d;\mu)\to L^2(\mathbb{R}^d;\mu)} \leq C(A,m,d,\tau)$.

Remarks about T(1)-Theorem for Bergman-Type Operators

This theorem gives an extension of the non-homogeneous harmonic analysis of Nazarov, Treil and Volberg to "Bergman-type" operators.

- The balls for which we have $\mu(B(x,r)) > r^m$ are called "non-Ahlfors balls".
  - Non-Ahlfors balls are enemies, their presence make the estimate of Calderón-Zygmund operator basically impossible.
  - The key hypothesis is that we can capture all the non-Ahlfors balls in some open set H.
  - This is just a restatement of the Carleson measure condition in this context.
- To handle this difficulty we suppose that our Calderón-Zygmund kernels have an additional estimate in terms of the behavior of the distance to the complement of H (namely that they are Bergman-type kernels).

Main Results

Characterization of Carleson Measures for $B_\sigma^2(\mathbb{B}_n)$

Theorem (Characterization of Carleson Measures for Besov-Sobolev Spaces, Volberg and Wick (2009))

Let μ be a positive Borel measure in $\mathbb{B}_n$. Then the following conditions are equivalent:

(a) μ *is a* $B_2^\sigma(\mathbb{B}_n)$*-Carleson measure;*

(b) $T_{\mu,2\sigma} : L^2(\mathbb{B}_n;\mu) \to L^2(\mathbb{B}_n;\mu)$ *is bounded;*

(c) *There is a constant* C *such that*

   (i) $\| T_{\mu,2\sigma}\chi_Q\|^2_{L^2(\mathbb{B}_n;\mu)} \leq C\,\mu(Q)$ *for all* Δ*-cubes* Q*;*

   (ii) $\mu(B_\Delta(x,r)) \leq C\, r^{2\sigma}$ *for all balls* $B_\Delta(x,r)$ *that intersect* $\mathbb{C}^n \setminus \mathbb{B}_n$*.*

Above, the sets B_Δ are balls measured with respect to the metric Δ and the set Q is a "cube" defined with respect to the metric Δ.

Remarks about Characterization of Carleson Measures

- We have already proved that $(a) \Leftrightarrow (b)$, and it is trivial $(b) \Rightarrow (c)$.
- It only remains to prove that $(c) \Rightarrow (b)$.
  - The proof of this Theorem follows from the T(1)-Theorem for Bergman-type operators.
  - In a neighborhood of the sphere $\partial\mathbb{B}_n$ the metric Δ looks a Euclidean-type quasi-metric. For example when $n = 2$ we have that

$$\Delta(x, y) \approx |x_1 - y_1| + |x_2 - y_2| + |x_3 - y_3|^2 + |x_4 - y_4|^2$$

  - The method of proof of the Euclidean Bergman-type T(1) theorem can then be modified to case of Calderón-Zygmund operators with respect to a quasi-metric (essentially verbatim).
- It is possible to show that the T(1) condition reduces to the simpler conditions in certain cases.
- An alternate proof of this Theorem was recently given by Hytönen and Martikainen. Their proof used a non-homogeneous T(b)-Theorem on metric spaces.

Sketch of Proof

Littlewood-Paley Decomposition

- Construct two independent dyadic lattices $\mathcal{D}_1$ and $\mathcal{D}_2$.
- We will call the cube $Q \in \mathcal{D}_i$ a *terminal cube* if $2Q$ is contained in our open set H or $\mu(Q) = 0$. All other cubes are called *transit* cubes. Denote by $\mathcal{D}_i^{term}$ and $\mathcal{D}_i^{tr}$ the terminal and transit cubes from $\mathcal{D}_i$.
- There are special unit cubes Q^0 and R^0 of $\mathcal{D}_1$ and $\mathcal{D}_2$ respectively that transit cubes and contain suppμ deep inside them.
- Define expectation operators Δ_Q (Haar function on Q) and Λ (average on Q^0), then we have for every $\varphi \in L^2(\mathbb{R}^d; \mu)$

$$\varphi = \Lambda\varphi + \sum_{Q\in\mathcal{D}_1^{tr}} \Delta_Q\varphi,$$

the series converges in $L^2(\mathbb{R}^d; \mu)$ and, moreover,

$$\|\varphi\|^2_{L^2(\mathbb{R}^d;\mu)} = \|\Lambda\varphi\|^2_{L^2(\mathbb{R}^d;\mu)} + \sum_{Q\in\mathcal{D}_1^{tr}} \|\Delta_Q\varphi\|^2_{L^2(\mathbb{R}^d;\mu)}.$$

Sketch of Proof

Good and Bad Cubes

For a dyadic cube R we denote $\cup_{i=1}^{2^d} \partial R_i$ by *sk* R, called the *skeleton* of R. Here the R_i are the dyadic children of R.

Definition (Bad Cubes)

Let τ, m be parameters of the Calderón-Zygmund kernel k. We fix $\alpha = \frac{\tau}{2\tau+2m}$. Fix a small number $\delta > 0$. Fix $S \geq 2$ to be chosen later. Choose an integer r such that

$$2^{-r} \leq \delta^S < 2^{-r+1}.$$

A cube $Q \in \mathcal{D}_1$ is called *bad* (or δ-bad) if there exists $R \in \mathcal{D}_2$ such that

- $\ell(R) \geq 2^r \ell(Q)$,
- $\mathrm{dist}(Q, sk\, R) < \ell(Q)^\alpha \ell(R)^{1-\alpha}$.

Sketch of Proof

Good and Bad Decomposition

We fix the decomposition of f and g into good and bad parts:

$$f = f_{good} + f_{bad}\,,\ \text{where } f_{good} = \Lambda f + \sum_{Q\in\mathcal{D}_1^{tr}\cap\mathcal{G}_1} \Delta_Q f$$

$$g = g_{good} + g_{bad}\,,\ \text{where } g_{good} = \Lambda g + \sum_{R\in\mathcal{D}_2^{tr}\cap\mathcal{G}_2} \Delta_R g.$$

One can choose $S = S(\alpha)$ in such a way that for any fixed $Q \in \mathcal{D}_1$,

$$\mathbb{P}\{Q \text{ is bad}\} \leq \delta^2$$

$$\mathbb{E}(\|f_{bad}\|_{L^2(\mathbb{R}^d;\mu)}) \leq \delta\|f\|_{L^2(\mathbb{R}^d;\mu)}.$$

Similar Statements for g hold as well.

Sketch of Proof

Reduction to Controlling The Good Part

- Using the decomposition above, we have

$$\langle T_{\mu,m} f, g\rangle_{L^2(\mathbb{R}^d;\mu)} = \langle T_{\mu,m} f_{good}, g_{good}\rangle_{L^2(\mathbb{R}^d;\mu)} + R(f,g)$$

- Using the construction above, we have that

$$\mathbb{E}|R_\omega(f,g)| \leq 2\,\delta \|T\|_{L^2(\mathbb{R}^d;\mu)\to L^2(\mathbb{R}^d;\mu)} \|f\|_{L^2(\mathbb{R}^d;\mu)} \|g\|_{L^2(\mathbb{R}^d;\mu)}.$$

- Choosing δ small enough ($< \frac{1}{4}$) we only need to show that

$$\left|\langle T_{\mu,m} f_{good}, g_{good}\rangle_{L^2(\mathbb{R}^d;\mu)}\right| \leq C(\tau, m, A, d) \|f\|_{L^2(\mathbb{R}^d;\mu)} \|g\|_{L^2(\mathbb{R}^d;\mu)}.$$

- This will then give

$$\|T\|_{L^2(\mathbb{R}^d;\mu)\to L^2(\mathbb{R}^d;\mu)} \leq 2C(\tau, m, A, d)$$

Sketch of Proof

Estimating The Good Part

- We then decompose the

$$\langle T_{\mu,m} f_{good}, g_{good} \rangle_{L^2(\mathbb{R}^d;\mu)} = A_1 + A_2 + A_3$$

- The term A_1 is the diagonal part of the sum. This is the easiest part.
- The term A_2 is the long-range interaction part. The second easiest part
  - Here we use the Calderón-Zygmund Estimates and the hypothesis that we can capture all the non-Ahlfors balls in the open set H.
- The term A_3 is the short-range interaction part.
  - Here we use the T(1) hypothesis and reduce the estimates to paraproducts.
- These all then imply that

$$\left| \langle T_{\mu,m} f_{good}, g_{good} \rangle_{L^2(\mathbb{R}^d;\mu)} \right| \leq C(\tau, m, A, d) \|f\|_{L^2(\mathbb{R}^d;\mu)} \|g\|_{L^2(\mathbb{R}^d;\mu)}.$$

Extension to "Nice" Metric Spaces

- It is possible to extend these results to Ahlfor regular metric spaces. Namely, (X, ρ, ν) with (X, ρ) a complete metric space and ν a Borel measure on X such that

$$\nu(B_r(x)) \approx r^n$$

- Suppose we have another measure μ on the metric space X (which need not be doubling), but satisfies the following relationship, for some $0 \leq m < n$

$$\mu\left(B(x,r)\right) \lesssim r^m \quad \forall x \in X, \quad \forall r. \tag{H}$$

- A *standard Calderón-Zygmund kernel of order* $0 < m \leq n$ is a function $k : X \times X \setminus \{x = y\} \to \mathbb{C}$ such that there exists constants C_{CZ}, $\tau, \delta > 0$

$$|k(x,y)| \leq \frac{C_{CZ}}{\rho(x,y)^m} \quad \forall x \neq y \in X;$$

Extension to "Nice" Metric Spaces

and

$$|k(x,y)-k(x,y')|+|k(x,y)-k(x',y)|\leq C_{CZ}\frac{\rho(x,x')^{\tau}}{\rho(x,y)^{\tau+m}}$$

provided that $\rho(x,x')\leq\delta\rho(x,y)$. In this situation, we say that the kernel k satisfies the standard estimates.

- If we additionally suppose the kernels that have the additional property that
$$|k(x,y)|\leq\frac{1}{\max(d^m(x),d^m(y))},$$
where $d(x):=\mathrm{dist}(x,X\setminus\Omega)=\inf\{\rho(x,y):y\in X\setminus\Omega\}$ and Ω being an open set in X.
- Then the analogue of the Bergman-type T(1) Theorem holds with essentially the same proof.

Thank You!