Carleson Measures for Hilbert Spaces of Analytic Functions

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Setup and General Overview

Let Ω be an open set in $\mathbb{C}^n$;
Let $\mathcal{H}$ be a Hilbert function space over Ω with reproducing kernel K_λ:

$$f(\lambda) = \langle f, K_\lambda \rangle_{\mathcal{H}} .$$

Definition ($\mathcal{H}$-Carleson Measure)

A non-negative measure μ on Ω is $\mathcal{H}$-Carleson if and only if

$$\int_\Omega |f(z)|^2 \, d\mu(z) \leq C(\mu)^2 \, \|f\|_{\mathcal{H}}^2 .$$

Question

Give a 'geometric' and 'testable' characterization of the $\mathcal{H}$-Carleson measures.

Obvious Necessary Conditions for Carleson Measures

Let k_λ denote the normalized reproducing kernel for the space $\mathcal{H}$:

$$k_\lambda(z) = \frac{K_\lambda(z)}{\|K_\lambda\|_{\mathcal{H}}}.$$

Testing on the reproducing kernel k_λ we always have a necessary geometric condition for the measure μ to be Carleson:

$$\sup_{\lambda\in\Omega} \int_\Omega |k_\lambda(z)|^2 \, d\mu(z) \leq C(\mu)^2.$$

In the cases of interest it is possible to identify a point $\lambda \in \Omega$ with an open set, I_λ on the boundary of Ω. A 'geometric' necessary condition is:

$$\mu\left(T(I_\lambda)\right) \lesssim \|K_\lambda\|_{\mathcal{H}}^{-2}.$$

Here $T(I_\lambda)$ is the 'tent' over the set I_λ in the boundary $\partial\Omega$.

Reasons to Care about Carleson Measures

- Bessel Sequences/Interpolating Sequences/Riesz Sequences: Given $\Lambda = \{\lambda_j\}_{j=1}^{\infty} \subset \Omega$ determine functional analytic basis properties for the set $\{k_{\lambda_j}\}_{j=1}^{\infty}$:
  - $\{k_{\lambda_j}\}_{j=1}^{\infty}$ is Bessel iff μ_Λ is $\mathcal{H}$-Carleson;
  - $\{\lambda_j\}_{j=1}^{\infty}$ is Interpolating iff μ_Λ is $\mathcal{H}$-Carleson and separated.
  - $\{k_{\lambda_j}\}_{j=1}^{\infty}$ is Riesz iff μ_Λ is $\mathcal{H}$-Carleson and separated.
- Multipliers of $\mathcal{H}$: Characterize the pointwise multipliers for $\mathcal{H}$:

$$\begin{aligned} \text{Multi}(\mathcal{H}) &= H^\infty \cap CM(\mathcal{H}). \\ \|\varphi\|_{\text{Multi}(\mathcal{H})} &\approx \|\varphi\|_{H^\infty} + \|\mu_\varphi\|_{CM(\mathcal{H})}. \end{aligned}$$

- Commutator/Bilinear Forms/Hankel Form Estimates: Given b, define $T_b : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$ by:

$$\begin{aligned} T_b(f,g) &= \langle fg, b\rangle_{\mathcal{H}}. \\ \|T_b\|_{\mathcal{H}\times\mathcal{H}\to\mathbb{C}} &\approx \|\mu_b\|_{CM(\mathcal{H})}. \end{aligned}$$

Choose your own talk?

While at a conference at Oberwolfach the audience discovers that it is interested in a certain class of Carleson measures for a Hilbert space of analytic functions. While looking into this question, the audience comes to a fork in the road and must choose which direction to proceed. Each direction has challenges, but miraculously in both directions the challenges can be overcome with similar tools! Which way do you choose....

- I am an analyst that cares more about several complex variables, function theory, Carleson measures, and their interaction.

  ▸ Characterization of Carleson Measures for Besov-Sobolev Space $B_2^\sigma(\mathbb{B}_n)$

- I am an analyst that cares mores about one complex variable, inner functions, Carleson measures, and their interaction.

  ▸ Characterization of Carleson Measures for the Model Space K_ϑ on $\mathbb{D}$

Talk Outline

- Motivation of the Problem
    - Besov-Sobolev Spaces of analytic functions on $\mathbb{B}_n$
    - Carleson Measures for Besov-Sobolev Spaces
    - Connections to Non-Homogeneous Harmonic Analysis
    - Characterization of Carleson measures for Besov-Sobolev Spaces

Besov-Sobolev Spaces

- The space $B_2^\sigma(\mathbb{B}_n)$ is the collection of holomorphic functions f on the unit ball $\mathbb{B}_n := \{z \in \mathbb{C}^n : |z| < 1\}$ such that

$$\left\{\sum_{k=0}^{m-1}\left|f^{(k)}(0)\right|^2 + \int_{\mathbb{B}_n}\left|\left(1-|z|^2\right)^{m+\sigma} f^{(m)}(z)\right|^2 d\lambda_n(z)\right\}^{\frac{1}{2}} < \infty,$$

  where $d\lambda_n(z) = \left(1-|z|^2\right)^{-n-1} dV(z)$ is the invariant measure on $\mathbb{B}_n$ and $m+\sigma > \frac{n}{2}$.
- Various choices of σ recover important classical function spaces:
  - $\sigma = 0$: Corresponds to the Dirichlet Space;
  - $\sigma = \frac{1}{2}$: Drury-Arveson Hardy Space;
  - $\sigma = \frac{n}{2}$: Classical Hardy Space;
  - $\sigma > \frac{n}{2}$: Bergman Spaces.

Besov-Sobolev Spaces

- The spaces $B_2^\sigma(\mathbb{B}_n)$ are reproducing kernel Hilbert spaces:

$$f(\lambda) = \langle f, K_\lambda^\sigma \rangle_{B_2^\sigma(\mathbb{B}_n)}$$

- A computation shows the kernel function $K_\lambda^\sigma(z)$ is:

$$K_\lambda^\sigma(z) = \frac{1}{\left(1 - \overline{\lambda} z\right)^{2\sigma}}$$

  - $\sigma = \frac{1}{2}$: Drury-Arveson Hardy Space; $K_\lambda^{\frac{1}{2}}(z) = \frac{1}{1-\overline{\lambda}z}$;
  - $\sigma = \frac{n}{2}$: Classical Hardy Space; $K_\lambda^{\frac{n}{2}}(z) = \frac{1}{(1-\overline{\lambda}z)^n}$;
  - $\sigma = \frac{n+1}{2}$: Bergman Space; $K_\lambda^{\frac{n+1}{2}}(z) = \frac{1}{(1-\overline{\lambda}z)^{n+1}}$.

Carleson Measures for Besov-Sobolev Spaces

We always have the following necessary condition:

$$\mu\left(T(B_r)\right) \lesssim r^{2\sigma}.$$

When $0 \leq \sigma \leq \frac{1}{2}$:

- If $n = 1$, the characterization can be expressed in terms of capacitary conditions. More precisely,

  $$\mu\left(T(G)\right) \lesssim \text{cap}_\sigma\left(G\right) \quad \forall \text{open } G \subset \mathbb{T}.$$

  See for example Stegenga, Maz'ya, Verbitsky, Carleson.
- If $n > 1$ then there are two different characterizations of Carleson measures for $B_2^\sigma(\mathbb{B}_n)$:
  - One method via dyadic tree structures on the ball by Arcozzi, Rochberg, and Sawyer.
  - One method via "T(1)" conditions by Tchoundja.

Question (Main Problem: Characterization in the Difficult Range)

Characterize the Carleson measures when $\frac{1}{2} < \sigma < \frac{n}{2}$.

Operator Theoretic Characterization of Carleson Measures

- A measure μ is Carleson exactly if the inclusion map ι from $\mathcal{H}$ to $L^2(\Omega;\mu)$ is bounded, or

$$\int_\Omega |f(z)|^2 \, d\mu(z) \leq C(\mu)^2 \, \|f\|_{\mathcal{H}}^2 .$$

Proposition

A measure μ is a $\mathcal{H}$-Carleson measure if and only if the linear map

$$T(f)(z) = \int_\Omega \operatorname{Re} K_x(z) \, f(x) \, d\mu(x)$$

is bounded on $L^2(\Omega;\mu)$.

Connections to Calderón-Zygmund Operators

When we apply this proposition to the spaces $B_2^{\sigma}(\mathbb{B}_n)$ this suggests that we study the operator

$$T_{\mu,2\sigma}(f)(z) = \int_{\mathbb{B}_n} \mathrm{Re}\left(\frac{1}{(1-\overline{w}z)^{2\sigma}}\right) f(w)\,d\mu(w) : L^2(\mathbb{B}_n;\mu) \to L^2(\mathbb{B}_n;\mu)$$

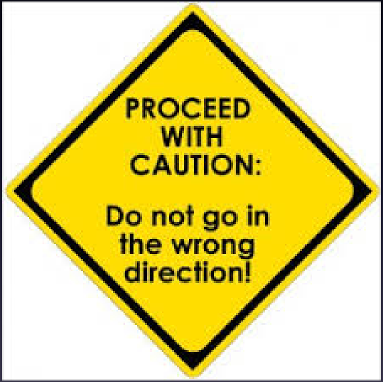
and find some conditions that will let us determine when it is bounded.

Tool that saves us: (Non-homogeneous) Harmonic Analysis.

- The kernel of the above integral operator has some cancellation and size estimates that are reminiscent of Calderón-Zygmund operators as living on a smaller dimensional space.
- The measure μ has a growth condition similar to the estimates on the kernel.
- Idea: Try to use T(1) to characterize the boundedness of

$$T_{\mu,2\sigma} : L^2(\mathbb{B}_n;\mu) \to L^2(\mathbb{B}_n;\mu).$$

Danger: Proof will Fail with out Careful Coordination!



Calderón–Zygmund Estimates for $T_{\mu,2\sigma}$

If we define

$$\Delta(z,w) := \begin{cases} \left||z|-|w|\right| + \left|1-\frac{z\overline{w}}{|z||w|}\right| & : \quad z,w \in \mathbb{B}_n \setminus \{0\} \\ |z|+|w| & : \quad \text{otherwise.} \end{cases}$$

Then Δ is a pseudo-metric and makes the ball into a space of homogeneous type.
A computation demonstrates that the kernel of $T_{\mu,2\sigma}$ satisfies the following estimates:

$$|K_{2\sigma}(z,w)| \lesssim \frac{1}{\Delta(z,w)^{2\sigma}} \quad \forall z,w \in \mathbb{B}_n;$$

If $\Delta(\zeta,w) < \frac{1}{2}\Delta(z,w)$ then

$$|K_{2\sigma}(\zeta,w) - K_{2\sigma}(z,w)| \lesssim \frac{\Delta(\zeta,w)^{1/2}}{\Delta(z,w)^{2\sigma+1/2}}.$$

Calderón-Zygmund Estimates for $T_{\mu,2\sigma}$

- These estimates on $K_{2\sigma}(z,w)$ say that it is a Calderón-Zygmund kernel of order 2σ with respect to the metric Δ.
  - Unfortunately, we can't apply the standard T(1) technology (adapted to a space of homogeneous type) to study the operators $T_{\mu,2\sigma}$. We would need the estimates of order n instead of 2σ.
- However, the measures we want to study (the Carleson measures for the space) satisfy the growth estimate

$$\mu\left(T(B_r)\right)\lesssim r^{2\sigma}$$

  and this is exactly the phenomenon that will save us!
- This places us in the setting of non-homogeneous harmonic analysis as developed by Nazarov, Treil, and Volberg. We have an operator with a Calderón-Zygmund kernel satisfying estimates of order 2σ, a measure μ of order 2σ, and are interested in $L^2(\mathbb{B}_n;\mu)\to L^2(\mathbb{B}_n;\mu)$ bounds.

Main Results

Theorem (Characterization of Carleson Measures for $B^2_\sigma(\mathbb{B}_n)$, Volberg and Wick, Amer. J. Math., 134 (2012))

Let μ be a positive Borel measure in $\mathbb{B}_n$. Then the following conditions are equivalent:

(a) *μ is a $B^\sigma_2(\mathbb{B}_n)$-Carleson measure;*

(b) *$T_{\mu,2\sigma} : L^2(\mathbb{B}_n;\mu) \to L^2(\mathbb{B}_n;\mu)$ is bounded;*

(c) *There is a constant C such that*

  (i) $\| T_{\mu,2\sigma}\chi_Q\|^2_{L^2(\mathbb{B}_n;\mu)} \leq C\,\mu(Q)$ *for all Δ-cubes Q;*

  (ii) $\mu(B_\Delta(x,r)) \leq C\,r^{2\sigma}$ *for all balls $B_\Delta(x,r)$ that intersect $\mathbb{C}^n \setminus \mathbb{B}_n$.*

Above, the sets B_Δ are balls measured with respect to the metric Δ and the set Q is a "cube" defined with respect to the metric Δ.

Remarks about Characterization of Carleson Measures

- We have already proved that $(a) \Leftrightarrow (b)$, and it is trivial $(b) \Rightarrow (c)$.
- It only remains to prove that $(c) \Rightarrow (b)$.
    - The proof of this theorem follows from a real variable proof of the T(1)-Theorem for Bergman-type operators.
    - Follow the proof strategy for the T(1) theorem in the context at hand. Technical but well established path (safe route!).
- It is possible to show that the T(1) condition reduces to the simpler conditions in certain cases.
- An alternate proof of this Theorem was later given by Hytönen and Martikainen. Their proof used a non-homogeneous T(b)-Theorem on metric spaces!

Safe Passage to the End!

Return to Beginning

Conclusion

Talk Outline

- Motivation of the Problem
    - Model Space of Functions on the Unit Disc
    - Carleson Measures for the Model Space
    - Connections to Two-Weight Inequalities for the Hilbert Transform
    - Characterization of Carleson measures for K_ϑ

The Model Space

Let H^2 denote the Hardy space on the unit disc $\mathbb{D}$;
Let ϑ denote an inner function on $\mathbb{D}$:

$$|\vartheta(\xi)| = 1 \quad \text{a.e. } \xi \in \mathbb{T}.$$

Let $K_\vartheta = H^2 \ominus \vartheta H^2$.
This is a reproducing kernel Hilbert space with kernel:

$$K_\lambda(z) = \frac{1 - \overline{\vartheta(\lambda)}\vartheta(z)}{1 - \overline{\lambda} z}.$$

Question (Carleson Measure Problem for K_ϑ)

Geometrically characterize the Carleson measures for K_ϑ:

$$\int_{\overline{\mathbb{D}}} |f(z)|^2 \, d\mu(z) \leq C(\mu)^2 \, \|f\|^2_{K_\vartheta} \quad \forall f \in K_\vartheta.$$

Carleson Measures for K_ϑ

We always have the necessary condition:

$$\int_{\mathbb{D}} \frac{\left|1-\overline{\vartheta(\lambda)}\vartheta(z)\right|^2}{\left|1-\overline{\lambda}z\right|^2} d\mu(z) \leq C(\mu)^2 \left\|K_\lambda\right\|_{K_\vartheta}^2 \quad \forall \lambda \in \mathbb{D}.$$

- If ϑ is a one-component inner function: Namely,

$$\Omega(\epsilon) \equiv \{z \in \mathbb{D} : |\vartheta(z)| < \epsilon\}, \qquad 0 < \epsilon < 1$$

  is connected for some ϵ.
    - Cohn proved that μ is a K_ϑ-Carleson measure if and only if the testing conditions hold for Carleson boxes that intersect $\Omega(\epsilon)$.
    - Treil and Volberg gave an alternate proof of this. Their proof works for $1 < p < \infty$.
- Nazarov and Volberg proved the obvious necessary condition is not sufficient for μ to be a K_ϑ-Carleson measure.

The Two-Weight Cauchy Transform

- Let σ denote a measure on $\mathbb{R}$.
- Let τ denote a measure on $\overline{\mathbb{R}^2_+}$.
- For $f \in L^2(\mathbb{R}, \sigma)$, the Cauchy transform will be

$$\mathsf{C}_\sigma(f)(z) = \int_{\mathbb{R}} \frac{f(w)}{w - z} \sigma(dw) = \mathsf{C}(\sigma f)(z).$$

- Let σ denote a measure on $\mathbb{T}$.
- Let τ denote a measure on $\overline{\mathbb{D}}$.
- For $f \in L^2(\mathbb{T}, \sigma)$, the Cauchy transform will be

$$\mathsf{C}_\sigma(f)(z) = \int_{\mathbb{T}} \frac{f(w)}{1 - \overline{w} z} \sigma(dw) = \mathsf{C}(\sigma f)(z).$$

Two-Weight Inequality for the Cauchy Transform

Theorem (Lacey, Sawyer, Shen, Uriarte-Tuero, W.)

Let σ be a weight on $\mathbb{T}$ and τ a weight on $\overline{\mathbb{D}}$. The inequality below holds, for some finite positive $\mathscr{C}$,

$$\|\mathsf{C}(\sigma f)\|_{L^2(\overline{\mathbb{D}};\tau)} \leq \mathscr{C}\|f\|_{L^2(\mathbb{T};\sigma)},$$

if and only if these constants are finite:

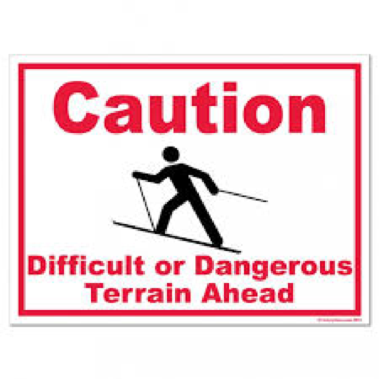
$$\sigma(\mathbb{T})\cdot\tau(\overline{\mathbb{D}}) + \sup_{z\in\mathbb{D}}\{\mathsf{P}(\sigma\mathbf{1}_{\mathbb{T}\setminus I})(z)\mathsf{P}\tau(z) + \mathsf{P}\sigma(z)\mathsf{P}(\tau\mathbf{1}_{\overline{\mathbb{D}}\setminus B_I})(z)\} \equiv \mathscr{A}_2,$$

$$\sup_I \sigma(I)^{-1}\int_{B_I}|\mathsf{C}_\sigma\mathbf{1}_I(z)|^2\tau(dA(z)) \equiv \mathscr{T}^2,$$

$$\sup_I \tau(B_I)^{-1}\int_I|\mathsf{C}_\tau^*\mathbf{1}_{B_I}(w)|^2\sigma(dw) \equiv \mathscr{T}^2.$$

Finally, we have $\mathscr{C} \simeq \mathscr{A}_2^{1/2} + \mathscr{T}$.

Danger: Technical Obstructions Exist!



Connection to Two-Weight Hilbert Transform

Recast the problem as a 'real-variable' question:

$$\mathsf{R}\sigma(x) \equiv \int_{\mathbb{R}} \frac{x-t}{|x-t|^2}\sigma(dt), \qquad x \in \mathbb{R}^2_+.$$

Write the coordinates of this operator as $(\mathsf{R}^1, \mathsf{R}^2)$. The second coordinate R^2 is the Poisson transform P. The Cauchy transform is

$$\mathsf{C}\sigma \equiv \mathsf{R}^1\sigma + i\mathsf{R}^2\sigma.$$

Question

Let σ denote a weight on $\mathbb{R}$ and τ denote a measure on the upper half plane $\mathbb{R}^2_+$. Find necessary and sufficient conditions on the pair of measures σ and τ so that the estimate below holds:

$$\|\mathsf{R}(\sigma f)\|_{L^2(\mathbb{R}^2_+;\tau)} \leq \mathscr{N}\|f\|_{L^2(\mathbb{R};\sigma)}.$$

Observations about the Problem

- The kernel of this operator:

$$\frac{x-t}{|x-t|^2}$$

is one-dimensional. Proofs and hypotheses should reflect this structure in some way.

- The necessity of the conditions is well-known:

$$\sigma(\mathbb{T})\cdot\tau(\overline{\mathbb{D}})+\sup_{z\in\mathbb{D}}\{\mathsf{P}(\sigma\mathbf{1}_{\mathbb{T}\setminus I})(z)\mathsf{P}\tau(z)+\mathsf{P}\sigma(z)\mathsf{P}(\tau\mathbf{1}_{\overline{\mathbb{D}}\setminus B_I})(z)\}\equiv\mathscr{A}_2,$$

$$\sup_I \sigma(I)^{-1}\int_{B_I}|\mathsf{C}_\sigma\mathbf{1}_I(z)|^2\tau(dA(z))\equiv\mathscr{T}^2,$$

$$\sup_I \tau(B_I)^{-1}\int_I|\mathsf{C}_\tau^*\mathbf{1}_{B_I}(w)|^2\sigma(dw)\equiv\mathscr{T}^2.$$

- All efforts have to go into showing the sufficiency of these conditions.
- Tool that saves us: (Non-homogeneous) Harmonic Analysis.

Main Ideas behind the Proof

- Use "hidden" positivity to deduce the 'Energy Inequality':

Lemma (Energy Inequality)

For any interval I_0 and partition $\mathcal{P}$ of I_0 into dyadic intervals,

$$\sum_{I\in\mathcal{P}}\sum_{K\in\mathcal{W}I} \mathsf{P}_\sigma(I_0\setminus K, K)^2 E(\tau, K)^2 \tau(Q_K) \lesssim \mathscr{R}^2\sigma(Q_{I_0}).$$

- Study the corresponding bilinear form:

$$\langle \mathsf{R}\sigma f, g\rangle_w$$

  and expand f and g with respect to weight Haar bases adapted to the weights σ and w.
- Mimic parts of the proof of the T1 Theorem. Use a clever recursion argument discovered in the case of the Hilbert transform.

Connecting the Cauchy Transform to Carleson Measures

- Let σ denote the Clark measure on $\mathbb{T}$.
- Then $L^2(\mathbb{T};\sigma)$ is unitarily equivalent to K_ϑ via a unitary U.
- $U^* : L^2(\mathbb{T};\sigma) \to K_\vartheta$ has the integral representation given by

$$U^*f(z) \equiv (1-\vartheta(z)) \int_{\mathbb{T}} \frac{f(\xi)}{1-\bar{\xi}z}\sigma(d\xi).$$

- For the inner function ϑ and measure μ, define a new measure $\nu_{\vartheta,\mu} \equiv |1-\vartheta|^2\mu$.

Lemma

A measure μ is a Carleson measure for K_ϑ if and only if $\mathsf{C} : L^2(\mathbb{T};\sigma) \to L^2(\overline{\mathbb{D}};\nu_{\vartheta,\mu})$ is bounded.

Characterization of Carleson Measures for K_ϑ

Theorem (Lacey, Sawyer, Shen, Uriarte-Tuero, W.)

Let μ be a non-negative Borel measure supported on $\overline{\mathbb{D}}$ and let ϑ be an inner function on $\mathbb{D}$ with Clark measure σ. Set $\nu_{\mu,\vartheta} = |1-\vartheta|^2\mu$. The following are equivalent:

(i) *μ is a Carleson measure for K_ϑ, namely,*

$$\int_{\overline{\mathbb{D}}} |f(z)|^2 \, d\mu(z) \leq C(\mu)^2 \, \|f\|_{K_\vartheta}^2 \quad \forall f \in K_\vartheta;$$

(ii) *The Cauchy transform C is a bounded map between $L^2(\mathbb{T};\sigma)$ and $L^2(\overline{\mathbb{D}};\nu_{\mu,\vartheta})$, i.e., $\mathsf{C} : L^2(\mathbb{T};\sigma) \to L^2(\overline{\mathbb{D}};\nu_{\vartheta,\mu})$ is bounded;*

(iii) *The three conditions in the above theorem hold for the pair of measures σ and $\nu_{\mu,\vartheta}$. Moreover,*

$$C(\mu) \simeq \|\mathsf{C}\|_{L^2(\mathbb{T};\sigma)\to L^2(\overline{\mathbb{D}};\nu_{\vartheta,\mu})} \simeq \mathscr{A}_2^{1/2} + \mathscr{T}.$$

Remarks about Characterization of Carleson Measures

- We have already seen that $(i) \Leftrightarrow (ii)$, and it is immediate $(ii) \Rightarrow (iii)$.
- It only remains to prove that $(iii) \Rightarrow (i)$.
    - The proof of this Theorem follows from a modification of the proof of the two-weight inequality for the Hilbert transform.
    - Follow the proof strategy as initiated by Nazarov, Treil, and Volberg. Use required modifications developed by Lacey, Sawyer, Shen, Uriarte-Tuero. Technical but established path (safe route!).
- It is possible to show that a similar characterization exists for d-dimensional Riesz transforms in $\mathbb{R}^n$. Full characterization is open still.

Safe Passage to the End!

Return to Beginning

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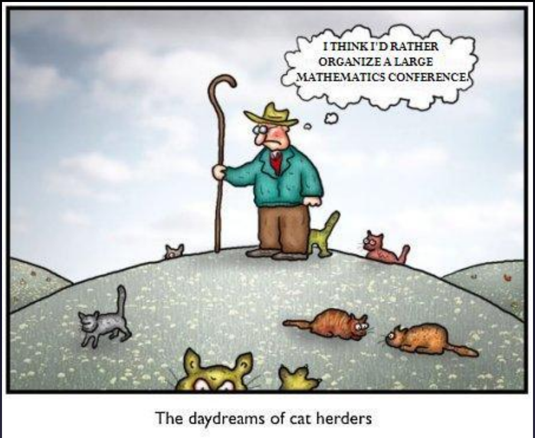

(Modified from the Original Dr. Fun Comic)

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Thank You!