The Essential Norm of Operators on the Bergman Space

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This talk is based on joint work with:

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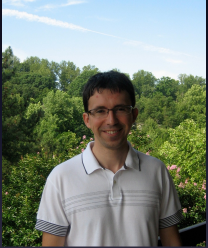

Mishko Mitkovski

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Weighted Bergman Spaces on $\mathbb{D}^n$

- Let $\mathbb{D}^n := \{z \in \mathbb{C}^n : |z_l| < 1, \quad l = 1, \ldots, n\}$.
- We let
$$dv(z) := \frac{1}{\pi^n} dA(z_1) \cdots dA(z_n).$$
- The space $A^2(\mathbb{D}^n)$ is the collection of holomorphic functions on $\mathbb{D}^n$ such that
$$\|f\|_{A^2(\mathbb{D}^n)}^2 := \int_{\mathbb{D}^n} |f(z)|^2 \, dv(z) < \infty.$$
- For $\lambda \in \mathbb{D}^n$ let $k_\lambda(z) := \prod_{l=1}^n \frac{(1-|\lambda_l|^2)}{(1-\overline{\lambda}_l z_l)^2}$ and $K_\lambda(z) := \prod_{l=1}^n \frac{1}{(1-\overline{\lambda}_l z_l)^2}$.
- Möbius maps: For any $z \in \mathbb{D}^n$ there exists a φ_z that interchanges 0 and z.

Weighted Bergman Spaces on $\mathbb{B}_n$

- Let $\mathbb{B}_n := \{z \in \mathbb{C}^n : |z| < 1\}$.
- For $\alpha > -1$, we let

$$dv_\alpha(z) := c_\alpha\,(1-|z|^2)^\alpha\,dv(z), \quad \text{with} \quad c_\alpha := \frac{\Gamma(n+\alpha+1)}{n!\,\Gamma(\alpha+1)}.$$

  The choice of c_α gives that $v_\alpha(\mathbb{B}_n) = 1$.
- The space $A^2_\alpha(\mathbb{B}_n)$ is the collection of holomorphic functions on $\mathbb{B}_n$ such that

$$\|f\|^2_{A^2_\alpha(\mathbb{B}_n)} := \int_{\mathbb{B}_n} |f(z)|^2\; dv_\alpha(z) < \infty.$$

- For $\lambda \in \mathbb{B}_n$ let $k^{(\alpha)}_\lambda(z) := \frac{(1-|\lambda|^2)^{\frac{n+1+\alpha}{2}}}{(1-\overline{\lambda}z)^{n+1+\alpha}}$ and $K_\lambda(z) := \frac{1}{(1-\overline{\lambda}z)^{n+1+\alpha}}$.
- Möbius maps: For any $z \in \mathbb{B}_n$ there exists a φ_z that interchanges 0 and z.

Fock Space on $\mathbb{C}^n$

- For $\beta > 0$ let $F^2_\beta(\mathbb{C}^n)$ be the collection of holomorphic functions on $\mathbb{C}^n$ for which
$$\|f\|^2_{F^2_\beta(\mathbb{C}^n)} := c_\beta \int_{\mathbb{C}^n} |f(z)|^2 \, e^{-\beta|z|^2} \, dv(z) < \infty.$$
- For $z \in \mathbb{C}^n$, let $K^\beta_z(w) := e^{\beta w \bar{z}}$ and $k^\beta_z(w) := e^{\beta w \bar{z} - \frac{\beta}{2}|z|^2}$.
- Möbius maps: For any $z \in \mathbb{C}^n$ there exists φ_z that interchanges 0 and z;
$$\varphi_z(w) := z - w.$$

Essential Axioms of the Previous Function Spaces

A.1 Let Ω be a domain (connected open set) in $\mathbb{C}^n$ containing the origin. Assume that for each $z \in \Omega$, there exists an involution $\varphi_z \in \mathrm{Aut}(\Omega)$ satisfying $\varphi_z(0) = z$.

A.2 There exists a metric d on Ω which is quasi-invariant under φ_z

$$d(u, v) \simeq d(\varphi_z(u), \varphi_z(v)) \quad \forall u, v \in \Omega.$$

A.3 There exists a finite Borel measure σ on Ω. Let the norm and inner product on $L^2(\Omega; d\sigma)$ be denoted by $\|\cdot\|$ and $\langle \cdot, \cdot \rangle$. $\mathcal{B}(\Omega) \subset L^2(\Omega; d\sigma)$ is the space of square integrable holomorphic functions on Ω. $\mathcal{B}(\Omega)$ is a RKHS with K_z and k_z the reproducing and the normalized reproducing kernel in $\mathcal{B}(\Omega)$.

Essential Axioms of the Previous Function Spaces

A.4 The measure $d\lambda(z) := \|K_z\|^2 \, d\sigma(z)$ is quasi-invariant under all φ_z and is a doubling measure;

$$\begin{aligned} \lambda(E) &\simeq \lambda(\varphi_z(E)) \quad E \subset \Omega; \\ \lambda(D(z,2r)) &\lesssim \lambda(D(z,r)) \quad \forall z \in \Omega,\ r > 0. \end{aligned}$$

A.5 For all $z, w \in \Omega$

$$|\langle k_z, k_w \rangle| \simeq \frac{1}{\left\|K_{\varphi_z(w)}\right\|}.$$

A.6 Rudin-Forelli Estimates: There exists κ with $0 \leq \kappa < 2$ such that

$$\int_\Omega \frac{|\langle K_z, K_w \rangle|^{\frac{r+s}{2}}}{\|K_z\|^s \|K_w\|^r} \, d\lambda(w) \leq C = C(r,s) < \infty, \ \ \forall z \in \Omega$$

for all $r > \kappa > s > 0$ or the above holds for all $r = s > 0$. In the latter case we will say that $\kappa = 0$.

A.7 The following limit holds: $\lim_{d(z,0)\to\infty} \|K_z\| = \infty$.

Sanity Check: $A^2(\mathbb{B}_n)$

Example (Bergman Spaces on $\mathbb{B}_n$)

A.1 $\varphi_z(w)$ *are simply the Möbius maps on* $\Omega = \mathbb{B}_n$*;*

A.2 $d(u,v)$ *is the hyperbolic metric on* $\mathbb{B}_n$*,* $d(u,v) := \frac{1}{2}\log\frac{1+|\varphi_u(v)|}{1-|\varphi_u(v)|}$*;*

A.3 $d\sigma$ *is normalized Lebesgue measure on* $\mathbb{B}_n$*.* $A^2(\mathbb{B}_n) \subset L^2(\mathbb{B}_n; d\sigma)$*.*

$K_z(w) := \frac{1}{(1-\overline{z}w)^{n+1}}$ *and* $k_z(w) := \frac{(1-|z|^2)^{\frac{n+1}{2}}}{(1-\overline{z}w)^{n+1}}$*;*

A.4 $d\lambda(z) := \frac{dv(z)}{(1-|z|^n)^{n+1}}$*;*

A.5 *Well-known "magic" identity:*

$$1-|\varphi_z(w)|^2 = \frac{(1-|z|^2)(1-|w|^2)}{|1-\overline{z}w|^2};$$

A.6 *Standard computation. Here* $\kappa = \frac{2n}{n+1}$*;*

A.7 $\lim_{|z|\to 1}\|K_z\|_{A^2} = +\infty$*, or* $k_z \rightharpoonup 0$ *as* $|z| \to 1$*.*

Sanity Check: $F^2(\mathbb{C}^n)$

Example (Fock Space on $\mathbb{C}^n$)

A.1 $\Omega = \mathbb{C}^n$ *and* $\varphi_z(w) := z - w$ *for each* $z \in \mathbb{C}^n$;

A.2 $d(u,v) = |u - v|$ *is the Euclidean metric on* $\mathbb{C}^n$;

A.3 $d\sigma(z) := \frac{1}{\pi} e^{-|z|^2} dv(z)$. $F^2(\mathbb{C}^n) \subset L^2(\mathbb{C}^n; d\sigma)$. $K_z(w) := e^{w\bar{z}}$ *and* $k_z(w) := e^{w\bar{z} - \frac{1}{2}|z|^2}$;

A.4 $d\lambda(z) := dv(z)$;

A.5 *Simple computation:*

$$e^{-\frac{1}{2}|z|^2 - \frac{1}{2}|w|^2 + \mathrm{Re}(\bar{z}w)} = e^{-\frac{1}{2}|z-w|^2};$$

A.6 *Standard computation integrating Gaussians. Here* $\kappa = 0$;

A.7 $\lim_{|z| \to +\infty} \|K_z\|_{F^2} = +\infty$.

The Berezin Transform

For $S \in \mathcal{L}(\mathcal{B}(\Omega))$ and $z \in \Omega$ we define the Berezin transform by

$$B(S)(z) := \langle Sk_z, k_z \rangle_{\mathcal{B}(\Omega)}.$$

- $B : \mathcal{L}(\mathcal{B}(\Omega)) \to L^\infty(\Omega)$:

$$|B(S)(z)| \leq \|S\|_{\mathcal{L}(\mathcal{B}(\Omega))} \|k_z\|_{\mathcal{B}(\Omega)} \|k_z\|_{\mathcal{B}(\Omega)} = \|S\|_{\mathcal{L}(\mathcal{B}(\Omega))}.$$

- If S is compact, then $B(S)(z) \to 0$ as $z \to \partial\Omega$:

$$|B(S)(z)| \leq \|Sk_z\|_{\mathcal{B}(\Omega)} \|k_z\|_{\mathcal{B}(\Omega)} = \|Sk_z\|_{\mathcal{B}(\Omega)}.$$

  However, $k_z \rightharpoonup 0$ as $z \to \partial\Omega$ and so $\|Sk_z\|_{\mathcal{B}(\Omega)} \to 0$.
- $B(S)$ is Lipschitz continuous with respect to the metric d

$$|B(S)(z_1) - B(S)(z_2)| \lesssim \|S\|_{\mathcal{L}(\mathcal{B}(\Omega))} d(z_1, z_2).$$

Toeplitz Operators and the Toeplitz Algebra

- The projection of $L^2(\Omega; d\sigma)$ onto $\mathcal{B}(\Omega)$ is given by the integral operator
$$P(f)(z) := \int_\Omega f(w) K_w(z)\, dv(w).$$
- Let M_a denote the operator of multiplication by the function a, $M_a(f) := af$. The Toeplitz operator with symbol $a \in L^\infty(\Omega)$ is the operator given by
$$T_a := PM_a.$$
- It is immediate to see that $\|T_a\|_{\mathcal{L}(\mathcal{B}(\Omega))} \lesssim \|a\|_{L^\infty(\Omega)}$.
- More generally, for a measure μ we will define the operator
$$T_\mu f(z) := \int_\Omega f(w) K_w(z)\, d\mu(w),$$
which will define an analytic function.

Toeplitz Operators and the Toeplitz Algebra

- For symbols in $L^\infty(\Omega)$ we let $\mathcal{T}_{L^\infty(\Omega)}$ be the C^* subalgebra of $\mathcal{L}(\mathcal{B}(\Omega))$ generated by T_a.
- An important class of operators in $\mathcal{T}_{L^\infty(\Omega)}$ are those that are finite sums of finite products of Toeplitz operators.

  Namely, for symbols $a_{jk} \in L^\infty(\Omega)$ with $1 \leq j \leq J$ and $1 \leq k \leq K$ we will need to study the operators:

$$\sum_{j=1}^{J} \prod_{k=1}^{K} T_{a_{jk}}$$

Question

Can we connect the behavior of the compact operators on $\mathcal{B}(\Omega)$, the Berezin transform, and the Toeplitz algebra?

Berezin Transform, Compactness, and Toeplitz Algebra

Theorem (Axler and Zheng, Indiana Univ. Math. J. 47 (1998))

Suppose that $a_{jk} \in L^\infty(\mathbb{D})$ with $1 \leq j \leq J$ and $1 \leq k \leq K$. Let $S = \sum_{j=1}^{J} \prod_{k=1}^{K} T_{a_{jk}}$ The following are equivalent:

(a) *The operator S is compact on $A^2(\mathbb{D})$;*

(b) $B(S)(z) \to 0$ *as* $|z| \to 1$;

(c) $\|Sk_z\|_{A^2(\mathbb{D})} \to 0$ *as* $|z| \to 1$.

Theorem (Engliš, Ark. Mat. 30 (1992))

If S is a compact operator on $A^2(\mathbb{B}_n)$, then $S \in \mathcal{T}_{L^\infty(\mathbb{B}_n)}$. True more generally for a domain $\Omega \subset \mathbb{C}^n$.

Characterization of Compact Operators on $A^2_\alpha(\mathbb{B}_n)$

Theorem (Suárez, Mitkovski and BDW, IEOT *to appear*)

Let $\alpha > -1$ and $S \in \mathcal{L}(A^2_\alpha(\mathbb{B}_n))$. Then S is compact if and only if $S \in \mathcal{T}_{L^\infty(\mathbb{B}_n)}$ and $\lim_{z\to\partial\mathbb{B}_n} B(S)(z) = 0$.

We can actually obtain much more precise information about the essential norm of an operator. For $S \in \mathcal{L}(A^2_\alpha(\mathbb{B}_n))$ recall that

$$\|S\|_e := \inf\left\{\|S-Q\|_{\mathcal{L}(A^2_\alpha(\mathbb{B}_n))} : Q \text{ is compact}\right\}.$$

We need to define other measures of the "size" of an operator $S \in \mathcal{L}(A^2_\alpha(\mathbb{B}_n))$:

$$\begin{aligned}
\mathfrak{a}_S &:= \lim_{r\to 1}\overline{\lim_{z\to\partial\mathbb{B}_n}}\sup\left\{\|Sf\|_{A^2_\alpha} : f \in T_{\mu 1_{D(z,r)}}(A^2_\alpha), \|f\|_{A^2_\alpha} \leq 1\right\};\\
\mathfrak{b}_S &:= \sup_{r>0}\limsup_{z\to\partial\mathbb{B}_n}\left\|M_{1_{D(z,r)}} S\right\|_{\mathcal{L}(A^2_\alpha(\mathbb{B}_n), L^2_\alpha(\mathbb{B}_n))};\\
\mathfrak{c}_S &:= \lim_{r\to 1}\left\|M_{1_{\mathbb{B}_n\setminus r\mathbb{B}_n}} S\right\|_{\mathcal{L}(A^2_\alpha(\mathbb{B}_n), L^2_\alpha(\mathbb{B}_n))}.
\end{aligned}$$

Characterization of Compact Operators on $A^2_\alpha(\mathbb{B}_n)$

Theorem (Suárez, Mitkovski and BDW, IEOT *to appear*)

Let $\alpha > -1$ and let $S \in \mathcal{T}_{L^\infty(\mathbb{B}_n)}$. Then there exists constants depending only on n and α such that:

$$\mathfrak{a}_S \approx \mathfrak{b}_S \approx \mathfrak{c}_S \approx \|S\|_e .$$

For the automorphism φ_z define the map $U_z^{(\alpha)} f(w) := f(\varphi_z(w)) k_z^{(\alpha)}(w)$. For $z \in \mathbb{B}_n$ and $S \in \mathcal{L}(A^2_\alpha(\mathbb{B}_n))$ we then define the map $S_z := U_z^{(\alpha)} S \left(U_z^{(\alpha)}\right)^*$.

Theorem (Suárez, Mitkovski and BDW, IEOT *to appear*)

Let $\alpha > -1$ and $S \in \mathcal{T}_{L^\infty(\mathbb{B}_n)}$. Then

$$\|S\|_e \approx \sup_{\|f\|_{A^2_\alpha(\mathbb{B}_n)}=1} \limsup_{z\to\partial\mathbb{B}_n} \|S_z f\|_{A^2_\alpha(\mathbb{B}_n)} .$$

Characterization of Compact Operators on $A^2(\mathbb{D}^n)$

Theorem (Mitkovski and BDW)

Let $S \in \mathcal{L}(A^2(\mathbb{D}^n))$*. Then* S *is compact if and only if* $S \in \mathcal{T}_{L^\infty(\mathbb{D}^n)}$ *and* $\lim_{z\to\partial\mathbb{D}^n} B(S)(z) = 0$*.*

Theorem (Mitkovski and BDW)

Let $S \in \mathcal{T}_{L^\infty(\mathbb{D}^n)}$*. Then there exists constants depending only on* n *such that:*

$$\mathfrak{a}_S \approx \mathfrak{b}_S \approx \mathfrak{c}_S \approx \|S\|_e .$$

Theorem (Mitkovski and BDW)

Let $S \in \mathcal{T}_{L^\infty(\mathbb{D}^n)}$*, then*

$$\|S\|_e \approx \sup_{\|f\|_{A^2(\mathbb{D}^n)}=1} \limsup_{z\to\partial\mathbb{D}^n} \|S_z f\|_{A^2(\mathbb{D}^n)} .$$

Characterization of Compact Operators on $F^2_\beta(\mathbb{C}^n)$

Theorem (Bauer and Isralowitz, J. Funct. Anal. 263 (2012))

Let $S \in \mathcal{L}(F^2_\beta(\mathbb{C}^n))$. Then S is compact if and only if $S \in \mathcal{T}_{L^\infty(\mathbb{C}^n)}$ and $\lim_{|z|\to\infty} B(S)(z) = 0$.

Theorem (Bauer and Isralowitz, J. Funct. Anal. 263 (2012))

Let $S \in \mathcal{T}_{L^\infty(\mathbb{C}^n)}$. Then there exists constants depending only on n and β such that:

$$\mathfrak{a}_S \approx \mathfrak{b}_S \approx \mathfrak{c}_S \approx \|S\|_e .$$

Theorem (Bauer and Isralowitz, J. Funct. Anal. 263 (2012))

Let $S \in \mathcal{T}_{L^\infty(\mathbb{C}^n)}$, then

$$\|S\|_e \approx \sup_{\|f\|_{F^2_\beta(\mathbb{C}^n)}=1} \limsup_{|z|\to\infty} \|S_z f\|_{F^2_\beta(\mathbb{C}^n)} .$$

Reproducing Kernel Thesis for Boundedness

For each $z \in \Omega$ we define an adapted translation operator U_z on $\mathcal{B}(\Omega)$ by

$$U_z f(w) := f(\varphi_z(w)) k_z(w).$$

Easy to show $\| U_z f\| \simeq \|f\|$ with implied constants independent of z. For any given operator T on $\mathcal{B}(\Omega)$ and $z \in \Omega$ we define $T_z := U_z T U_z^*$.

Theorem (Mitkovski and BDW)

Let $T : \mathcal{B}(\Omega) \to \mathcal{B}(\Omega)$ be a linear operator defined a priori only on the linear span of the normalized reproducing kernels of $\mathcal{B}(\Omega)$. Define T^* on the same set by duality. Let κ be the constant from A.6. If

$$\sup_{z\in\Omega} \| U_z T k_z\|_{L^p(\Omega;d\sigma)} < \infty \quad \textit{and} \quad \sup_{z\in\Omega} \| U_z T^* k_z\|_{L^p(\Omega;d\sigma)} < \infty$$

for some $p > \frac{4-\kappa}{2-\kappa}$ then T is bounded on $\mathcal{B}(\Omega)$.

Idea of Proof

$$\|Tf\|^2 \leq \int_\Omega \left| \int_\Omega |\langle K_w, T^* K_z \rangle| \, |f(w)| \, d\sigma(w) \right|^2 d\sigma(z).$$

Define $Rf(z) := \int_\Omega |\langle T^* K_z, K_w \rangle| \, f(w) d\sigma(w)$. It is enough to show that this operator is bounded on $L^2(\Omega; d\sigma)$. One then shows:

$$\int_\Omega R(z,w) \|K_w\|^\alpha \; d\sigma(w) \lesssim \sup_{z \in \Omega} \|U_z T^* k_z\|_{L^p(\Omega; d\sigma)} \|K_z\|^\alpha$$

$$\int_\Omega R(z,w) \|K_z\|^\alpha \; d\sigma(z) \lesssim \sup_{z \in \Omega} \|U_z T k_z\|_{L^p(\Omega; d\sigma)} \|K_w\|^\alpha .$$

Here choose $\alpha \in \left(\frac{2}{p}, \frac{4-2\kappa}{4-\kappa}\right)$ such that $q\left(\alpha - \frac{2}{p}\right) < \kappa$. The condition $p > \frac{4-\kappa}{2-\kappa}$ guarantees α exists. Schur's Test then shows R, and hence T is bounded.

Applications of Boundedness Result

Corollary (Mitkovski and BDW)

If there exists $p > \frac{4-\kappa}{2-\kappa}$ such that

$$\sup_{z \in \Omega} |g(z)|^p \int_\Omega |f(\varphi_z(w))|^p \, d\sigma(w) < \infty,$$

and

$$\sup_{z \in \Omega} |f(z)|^p \int_\Omega |g(\varphi_z(w))|^p \, d\sigma(w) < \infty,$$

then the operator $T_f T_{\bar{g}}$ is bounded on $\mathcal{B}(\Omega)$.

- Related to a famous conjecture of Sarason about the boundedness of the product $T_f T_{\bar{g}}$ on $A^2(\mathbb{D})$:

$$\sup_{z \in \mathbb{D}} \int_{\mathbb{D}} |f(\varphi_z(w))|^2 \, dA(w) \int_{\mathbb{D}} |g(\varphi_z(w))|^2 \, dA(w) < \infty.$$

Applications of Boundedness Result

Corollary (Mitkovski and BDW)

If $\kappa > 0$ *and* T_u *is a Toeplitz operator whose symbol* u *for some* $p > \frac{4-\kappa}{2-\kappa}$ *satisfies*

$$\sup_{z\in\Omega}\int_{\Omega} |u(\varphi_z(w))|^p \, d\sigma(w) < \infty,$$

then T_u *is bounded on* $\mathcal{B}(\Omega)$*.*

Corollary (Mitkovski and BDW)

If $\kappa > 0$ *and* $H_{\bar{f}}$ *is a Hankel operator whose symbol* $\bar{f}$ *satisfies*

$$\sup_{z\in\Omega}\int_{\Omega} |f(\varphi_z(w)) - f(z)|^p \, d\sigma(w) < \infty,$$

for some $p > \frac{4-\kappa}{2-\kappa}$ *then* $H_{\bar{f}}$ *is bounded.*

Reproducing Kernel Thesis for Compactness

Theorem (Mitkovski and BDW)

Let $T : \mathcal{B}(\Omega) \to \mathcal{B}(\Omega)$ *be a linear operator and* κ *be the constant from A.6. If*

$$\sup_{z\in\Omega} \|U_z T k_z\|_{L^p(\Omega;d\sigma)} < \infty \quad \textit{and} \quad \sup_{z\in\Omega} \|U_z T^* k_z\|_{L^p(\Omega;d\sigma)} < \infty,$$

for some $p > \frac{4-\kappa}{2-\kappa}$*, then*

(a) $\|T\|_e \simeq \sup_{\|f\|\leq 1} \limsup_{d(z,0)\to\infty} \|T_z f\|$.

(b) *If* $\lim_{d(z,0)\to\infty} \|Tk_z\| = 0$ *then* T *must be compact.*

Corollary (Mitkovski and BDW)

Let $\mathcal{B}(\Omega)$ *be Bergman-type space for which* $\kappa > 0$*. If* T *is in the Toeplitz algebra* $\mathcal{T}_{L^\infty(\Omega)}$ *then* $\|T\|_e \simeq \sup_{\|f\|\leq 1} \limsup_{d(z,0)\to\infty} \|T_z f\|$.

Idea of Proof

Proposition (Covering Lemma)

There exists an integer $N > 0$ such that for any $r > 0$ there is a covering $\mathcal{F}_r = \{F_j\}$ of Ω by disjoint Borel sets satisfying

(1) *every point of Ω belongs to at most N of the sets $G_j := \{z \in \Omega : d(z, F_j) \leq r\}$;*

(2) $\operatorname{diam}_d F_j \leq 2r$ *for every j.*

Lemma

Let (X, d) be a separable metric space and $r > 0$. There is a denumerable set of points $\{x_j\}$ and Borel subsets $\{Q_j\}$ of X that satisfy

(1) $X = \bigcup_j Q_j$;

(2) $Q_j \cap Q_{j'} = \emptyset$ *for $j \neq j'$;*

(3) $D(x_j, r) \subset Q_j \subset D(x_j, 2r)$.

Approximation Result

Proposition

Let $T : \mathcal{B}(\Omega) \to \mathcal{B}(\Omega)$ *be a linear operator and* κ *be the constant from A.6. If*

$$\sup_{z\in\Omega} \| U_z T k_z \|_{L^p(\Omega;d\sigma)} < \infty \quad \textit{and} \quad \sup_{z\in\Omega} \| U_z T^* k_z \|_{L^p(\Omega;d\sigma)} < \infty$$

for some $p > \frac{4-\kappa}{2-\kappa}$, *then for every* $\epsilon > 0$ *there exists* $r > 0$ *such that for the covering* $\mathcal{F}_r = \{F_j\}$

$$\left\| TP - \sum_j M_{1_{F_j}} TPM_{1_{G_j}} \right\|_{\mathcal{L}(\mathcal{B}(\Omega))} < \epsilon.$$

Allows us to obtain compact operators that approximate TP in norm.

Proof of Compactness Result

Since $\lim_{d(z,0)\to\infty} \|A_z f\| = 0$ for every compact operator A we obtain that

$$\sup_{\|f\|\leq 1} \limsup_{d(z,0)\to\infty} \|T_z f\| \leq \sup_{\|f\|\leq 1} \limsup_{d(z,0)\to\infty} \|(T-A)_z f\| \lesssim \|T-A\|.$$

But, since A is arbitrary this immediately implies

$$\sup_{\|f\|\leq 1} \limsup_{d(z,0)\to\infty} \|T_z f\| \lesssim \|T\|_e.$$

For the other direction:

- $\|T\|_e \approx \|TP\|_e$;
- Using the approximation result it suffices to show

$$\limsup_{m\to\infty} \left\| \sum_{j\geq m} M_{1_{F_j}} TPM_{1_{G_j}} \right\|_{\mathcal{L}(\mathcal{B}(\Omega), L^2(\Omega; d\sigma))} \lesssim \sup_{\|f\|\leq 1} \limsup_{d(z,0)\to\infty} \|T_z f\|_{\mathcal{L}(\mathcal{B}(\Omega))}$$

- One then mimics the proofs for $\mathbb{D}^n$, $\mathbb{B}_n$ or $\mathbb{C}^n$ (since they are all the same proof!).

The daydreams of cat herders

(Modified from the Original Dr. Fun Comic)

Thanks to Jaydeb and Raja for Organizing the Workshop!

Thank You!