Commutators and BMO

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Riesz Transforms, BMO and the Hardy Space H^1

For $1 \leq j \leq n$ let $R_j(f)(x) = c_n \int_{\mathbb{R}^n} \frac{x_j - y_j}{|x-y|^{n+1}} f(y)\, dy$ denote the Riesz transform in the jth variable.

Definition (Bounded Mean Oscillation)

$$\|b\|_{BMO(\mathbb{R}^n)} := \sup_Q \left(\frac{1}{|Q|} \int_Q |b(x) - b_Q|^2 \, dx \right)^{\frac{1}{2}}$$

Definition (Hardy Space)

$$H^1(\mathbb{R}^n) = \{ f \in L^1(\mathbb{R}^n) : R_j f \in L^1(\mathbb{R}^n) \}$$

$$\|f\|_{H^1(\mathbb{R}^n)} := \|f\|_{L^1(\mathbb{R}^n)} + \sum_{j=1}^n \|R_j f\|_{L^1(\mathbb{R}^n)} .$$

Commutators and BMO

Theorem (C. Fefferman (1971))

The dual of $H^1(\mathbb{R}^n)$ is $BMO(\mathbb{R}^n)$, i.e., $\left(H^1(\mathbb{R}^n)\right)^* = BMO(\mathbb{R}^n)$.

For each $j = 1, \ldots, n$ define the following commutator operator on $L^2(\mathbb{R}^n)$:

$$[b, R_j](f)(x) := b(x)R_j(f)(x) - R_j(bf)(x).$$

Theorem (Coifman, Rochberg, and Weiss (1976))

Let $b \in BMO(\mathbb{R}^n)$, then for $j = 1, \ldots, n$

$$\|[b, R_j]\|_{L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n)} \lesssim \|b\|_{BMO(\mathbb{R}^n)}.$$

If $\|[b, R_j]\|_{L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n)} < +\infty$ for $j = 1, \ldots, n$, then

$$\|b\|_{BMO(\mathbb{R}^n)} \lesssim \max_j \|[b, R_j]\|_{L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n)}.$$

Factorization and H^1

Define the following bilinear operators on $L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ by:

$$\Pi_j(g,h) = gR_jh + hR_jg \quad j = 1,\dots,n.$$

Theorem (Coifman, Rochberg, and Weiss (1976))

Let $f, g \in L^2(\mathbb{R}^n)$ then for $j = 1,\dots,n$:

$$\|\Pi_j(f,g)\|_{H^1(\mathbb{R}^n)} \lesssim \|f\|_{L^2(\mathbb{R}^n)} \|g\|_{L^2(\mathbb{R}^n)}.$$

Moreover, for any $f \in H^1(\mathbb{R}^n)$ there exists $g_k^j, h_k^j \in L^2(\mathbb{R}^n)$ so that $f = \sum_{j=1}^n \sum_{k=1}^\infty \Pi_j(g_k^j, h_k^j)$. And

$$\|f\|_{H^1(\mathbb{R}^n)} \approx \inf \left\{ \sum_{j=1}^n \sum_{k=1}^\infty \left\|g_k^j\right\|_{L^2(\mathbb{R}^n)} \left\|h_k^j\right\|_{L^2(\mathbb{R}^n)} : f = \sum_{j=1}^n \sum_{k=1}^\infty \Pi_j(g_k^j, h_k^j) \right\}.$$

Reasons to Care about These Results

1. The commutator $[b, H]$ (H Hilbert transform) connects to complex analysis. The Commutator Theorem is a reformulation of Nehari's Theorem and the factorization result is weakening of the strong factorization of analytic Hardy spaces.
   - $L^2 = H^2 \oplus (H^2)^\perp$ and we have that $\mathbb{P}_+$ is the Cauchy projection onto H^2.
   - The Hankel operator with symbol φ is the map from H^2 to $(H^2)^\perp$ and is defined as $h_\varphi(f) = (I - \mathbb{P}_+)(\varphi f) = [\varphi, \mathbb{P}_+](f)$.
   - $[b, H] = h_b - h^*_{\bar{b}}$
2. The Commutator Theorem says things about div-curl lemmas. If $\vec{B}$ and $\vec{E}$ are vector fields in L^2 with $\operatorname{curl} \vec{B} = 0$ and $\operatorname{div} \vec{E} = 0$ then we have that $\vec{E} \cdot \vec{B} \in H^1$.
   - $\vec{B}$ curl-free implies there exists a function $\varphi \in L^2(\mathbb{R}^n)$ such that $B_j = R_j\varphi$ and $\|B\|_{L^2(\mathbb{R}^n;\mathbb{R}^n)} \approx \|\varphi\|_{L^2(\mathbb{R}^n)}$.
   - $\vec{E}$ is divergence-free and so $\sum_{j=1}^n R_j E_j(x) = 0$;
   - $\vec{E} \cdot \vec{B}(x) = \sum_{j=1}^n E_j(x) B_j(x) = \sum_{j=1}^n E_j(x) R_j \varphi(x) + \varphi(x) R_j E_j(x)$.

Possible Generalizations

1. Change the Target and Domain Spaces:
   - Characterize the symbols b so that $[b, T] : L^p(X, \lambda_1) \to L^q(X, \lambda_2)$.
2. Change the Differential Operator you care about:
   - Can we characterize a BMO space for Riesz transforms $\nabla L^{-\frac{1}{2}}$ associated to operators L other than the Laplacian?
3. Change the geometry of the operator and underlying space:
   - Can we characterize the commutators when the operators are invariant under different dilation structures?
   - Can we characterize the commutators when they are defined on objects that possess certain geometric properties?

Also interested in combinations of the above questions.

Possible Proof Strategies

1. Proving the Upper bound:
   - Good λ inequalities.
   - Dyadic Harmonic Analysis Methods (paraproducts, shift operators).
   - Sparse Operators and Domination.
   - Cauchy Integral Trick.
2. Proving the Lower Bound:
   - Direct Testing of the Operator.
   - Uchiyama's Algorithm.

For some specific operators we have proofs we can exploit using the structure of the operator.

The Cauchy Integral Trick

Consider the operator: $S_z(f) = e^{\frac{zb}{2}} T(e^{-\frac{zb}{2}} f)$, where f is a "nice" function and z is a parameter related to some information about b. Expand in a power series in z and observe that:

$$\frac{d}{dz} S_z(f)\Big|_{z=0} = \frac{1}{4}[b, T](f).$$

The function $z \mapsto S_z(f)$ is holomorphic and so by the Cauchy Integral Formula we have:

$$\frac{1}{4}[b, T](f) = \frac{d}{dz} S_z(f)\Big|_{z=0} = \frac{4}{2\pi i} \int_{|z|=\epsilon} \frac{e^{\frac{zb}{2}} T(e^{-\frac{zb}{2}} f)}{z^2} dz.$$

The Cauchy Integral Trick

Two important facts are needed to take advantage of this computation:

Lemma

If $b \in BMO$, $|z| \le \epsilon \approx \frac{1}{\|b\|_{BMO}}$ *then* $e^{zb} \in A_2$ *with* $[e^{zb}]_{A_2} \lesssim 1$.

Theorem

If T *is a Calderón-Zygmund operator and* $w \in A_2$ *then*

$$\left\| T : L^2(w) \to L^2(w) \right\| \lesssim C(w, T).$$

The Cauchy Integral Trick

From this we have:

$$\begin{aligned}
\|[b,T](f)\|_{L^2} &= \left\| \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{e^{\frac{zb}{2}} T(e^{-\frac{zb}{2}} f)}{z^2} dz \right\|_{L^2} \\
&\lesssim \int_{|z|=\epsilon} \frac{\left\| e^{\frac{zb}{2}} T(e^{-\frac{zb}{2}} f) \right\|_{L^2}}{|z|^2} d\,|z| \\
&\lesssim \frac{\left\| e^{\frac{zb}{2}} T e^{-\frac{zb}{2}} : L^2 \to L^2 \right\|}{\epsilon} \|f\|_{L^2} \\
&= \frac{\sup_{|z|=\epsilon} \left\| T : L^2(e^{zb}) \to L^2(e^{zb}) \right\|}{\epsilon} \|f\|_{L^2} \\
&\lesssim \|b\|_{BMO} \|f\|_{L^2}.
\end{aligned}$$

So the commutator $[b,T] : L^2 \to L^2$ is bounded and the norm is controlled by the BMO norm of b.

Dyadic Harmonic Analysis Proof

Useful facts for this proof:

Theorem

If T is a Calderón-Zygmund operator, then T has a decomposition in terms of Haar shift operators:

$$T = \sum_{r,s} S_{r,s}.$$

Here

$$S_{r,s} f = \sum_{I \in \mathcal{D}} \sum_{J \in C_r(I)} \sum_{K \in C_s(I)} a_{I,J,K} \langle f, h_J \rangle h_K.$$

Theorem

If $b \in BMO$ then the paraproduct $\pi_b : L^2 \to L^2$ with

$$\left\| \pi_b : L^2 \to L^2 \right\| \lesssim \|b\|_{BMO}.$$

Dyadic Harmonic Analysis Proof

Also important is the following (paraproduct) decomposition:

$$bg = \pi_b g + \pi_b^* g + \pi_g b.$$

Since we can recover any operator T by Haar shifts, we can just study $[b, S_{r,s}]$ and obtain good estimates there. Observe now that for any operator S that we have by the decomposition:

$$\begin{aligned}
[b, S]f &= bSf - S(bf) \\
&= \pi_b Sf + \pi_b^* Sf + \pi_{Sf} b - S\left(\pi_b f + \pi_b^* f + \pi_f b\right) \\
&= \left(\pi_b S - S\pi_b\right) f + \left(\pi_b^* S - S\pi_b^*\right) f + (\pi_{Sf} - S\pi_f)b.
\end{aligned}$$

The first two terms are easy and give the estimate we want. The second term is an "error" but is amenable to direct analysis and computation since we are working with dyadic operators.

Uchiyama's Algorithm

Instead of proving $\|b\|_{BMO} \lesssim \|[b,T] : L^2 \to L^2\|$ directly, by duality it is enough to prove the factorization of H^1 directly. A function a is an atom if it is supported in an interval I, $\int_I a dx = 0$, and $\|a\|_{L^\infty} \leq \frac{1}{|I|}$.

Theorem (Atomic Decomposition)

Any $f \in H^1$ can be written via an atomic decomposition:
$f = \sum_{k=1}^{\infty} \alpha_k a_k$ *where a_k are atoms and* $\|f\|_{H^1} \approx \inf\{\sum_k |\alpha_k|\}$

Lemma (Splitting Atoms)

Let $\Pi_T(g,h) = gTh - hT^*g$. For any $\epsilon > 0$ and for all atoms a there exists $g, h \in L^2$ such that:

$$\begin{aligned} \|a - \Pi_T(g,h)\|_{H^1} &< \epsilon \\ \|g\|_{L^2} \|h\|_{L^2} &\leq C(\epsilon). \end{aligned}$$

Uchiyama's Algorithm

One then combines the atomic decomposition with slitting atoms to get the weak factorization.

$$\begin{aligned} f &= \sum_k \alpha_k^{(1)} a_k^{(1)} \\ &= \sum_k \alpha_k^{(1)} (a_k^{(1)} - \Pi_T(g_k^{(1)}, h_k^{(1)})) + \sum_k \alpha_k \Pi_T(g_k^{(1)}, h_k^{(1)}) \\ &= E_1 + M_1. \end{aligned}$$

We then have that:

$$\|E_1\|_{H^1} = \left\| \sum_k \alpha_k^{(1)} (a_k^{(1)} - \Pi_T(g_k^{(1)}, h_k^{(1)})) \right\|_{H^1} \leq C_a \epsilon \|f\|_{H^1}.$$

We can then apply the atomic decomposition to the function $\sum_k \alpha_k^{(1)} (a_k^{(1)} - \Pi_T(g_k^{(1)}, h_k^{(1)}))$ and have:

$$E_1 = \sum_k \alpha_k^{(2)} a_k^{(2)}.$$

Uchiyama's Algorithm

We can then apply the atomic decomposition to the function $E_1 = \sum_k \alpha_k^{(1)}(a_k^{(1)} - \Pi_T(g_k^{(1)}, h_k^{(1)}))$ and have:

$$\begin{aligned} E_1 &= \sum_k \alpha_k^{(2)} a_k^{(2)} \\ &= \sum_k \alpha_k^{(2)}(a_k^{(2)} - \Pi_T(g_k^{(2)}, h_k^{(2)})) + \sum_k \alpha_k \Pi_T(g_k^{(2)}, h_k^{(2)}) \\ &= E_2 + M_2. \end{aligned}$$

Again we then have:

$$\|E_2\|_{H^1} = \left\| \sum_k \alpha_k^{(2)}(a_k^{(2)} - \Pi_T(g_k^{(2)}, h_k^{(2)})) \right\|_{H^1} \leq C_a\epsilon \|E_1\|_{H^1} \leq (C_a\epsilon)^2 \|f\|_H$$

We can the choose that $C_a\epsilon < 1$ and iterate to get that $E_l \to 0$ and $f = \sum_l M_l$, which is the decomposition we want.

Commutators Acting Between Weighted Spaces

Definition

Let w be a weight on $\mathbb{R}^n$, i.e. w is an almost everywhere positive, locally integrable function. Set $w(Q) = \int_Q w(x)\,dx$ and $\langle w\rangle_Q = \frac{w(Q)}{|Q|}$. Then we say that w belongs to the Muckenhoupt class of A_p weights for some $1 < p < \infty$ provided that:

$$[w]_{A_p} = \sup_Q \langle w\rangle_Q \left\langle w^{1-q}\right\rangle_Q^{p-1} < \infty,$$

Theorem (Holmes, Lacey, W., Math. Ann. (2017))

For $1 < p < \infty$, and $\lambda_1, \lambda_2 \in A_p$, set $\nu = \lambda_1^{\frac{1}{p}}\lambda_2^{-\frac{1}{p}}$. Then there are constants $0 < c < C < \infty$, depending only on n, p, λ_1 and λ_2, for which

$$c\|b\|_{BMO_\nu(\mathbb{R}^n)} \leq \sum_{i=1}^n \left\|[b, R_i] : L^p_{\lambda_1}(\mathbb{R}^n) \to L^p_{\lambda_2}(\mathbb{R}^n)\right\| \leq C\|b\|_{BMO_\nu(\mathbb{R}^n)}.$$

Bloom's Theorem in Spaces of Homogeneous Type

- Let (X, d, μ) be a space of homogeneous type; i.e. d is a quasi metric and μ is a doubling measure.
- T is a Calderón–Zygmund operator on (X, d, μ) if T is bounded on $L^2(X)$ and has the associated kernel $K(x, y)$ such that $T(f)(x) = \int K(x, y) f(y) d\mu(y)$ for any $x \notin \operatorname{supp} f$, and $K(x, y)$ satisfies the following estimates: for all $x \neq y$,

$$|K(x, y)| \leq \frac{C}{V(x, y)},$$

and for $d(x, x') \leq (2A_0)^{-1} d(x, y)$,

$$|K(x, y) - K(x', y)| + |K(y, x) - K(y, x')| \leq \frac{C}{V(x, y)} \Big(\frac{d(x, x')}{d(x, y)}\Big)^{\eta}.$$

Here $V(x, y) = \mu(B(x, d(x, y)))$ and by the doubling condition we have that $V(x, y) \approx V(y, x)$.

Bloom in Spaces of Homogeneous Type

Definition

A function $f \in L^1_{\text{loc}}(X)$ belongs to $BMO_w(X)$ if

$$\|b\|_{BMO_w(X)} := \sup_Q \frac{1}{w(Q)} \int_Q |b(x) - b_Q| \, d\mu(x) < \infty.$$

Theorem (Duong, Gong, Kuffner, Li, W., 2017)

Suppose $1 < p < \infty$, $\lambda_1, \lambda_2 \in A_p$ *and* $\nu = \lambda_1^{\frac{1}{p}} \lambda_2^{-\frac{1}{p}}$ *and* $b \in \text{BMO}_\nu(X)$. *Then*

$$\|[b, T] : L^p_{\lambda_1}(X) \to L^p_{\lambda_2}(X)\| \lesssim \|b\|_{\text{BMO}_\nu(X)}.$$

(Partial) Converse to Bloom

Let M be a large positive number. For any fixed ball $B(x_0, r)$ centered at $x_0 \in X$ with radius $r > 0$ there exists a ball $B(y_0, r)$ centered at $y_0 \in X$ with radius $r > 0$ satisfying $d(x_0, y_0) > Mr$, such that T satisfies that for every $x \in B(x_0, r)$,

$$|T(\chi_{B(y_0,r)})(x)| \gtrsim \frac{\mu(B(y_0, r))}{V(x_0, y_0)}.$$

Theorem (Duong, Gong, Kuffner, Li, W., 2017)

Suppose $1 < p < \infty$, $\lambda \in A_p$. *Suppose that* T *is a Calderón-Zygmund operator that satisfies the condition above. Also suppose that* $[b, T]$ *is bounded from* $L^p_\lambda(X)$ *to* $L^p_\lambda(X)$. *Then* b *is in* $\mathrm{BMO}(X)$, *and*

$$\|b\|_{\mathrm{BMO}(X)} \lesssim \|[b, T] : L^p_\lambda(X) \to L^p_\lambda(X)\|.$$

The Bessel Operator

- Let $\mathbb{R}_+ = (0, \infty)$ and define the measure $dm_\lambda := x^{2\lambda} dx$ $(\lambda > 0)$. This is a space of homogeneous type.
- The Bessel operator is defined by

$$\Delta_\lambda f(x) := -\frac{d^2}{dx^2} f(x) - \frac{2\lambda}{x}\frac{d}{dx} f(x).$$

(Note we have absorbed the minus sign into the definition).
- One can show that this operator is non-negative and self-adjoint on $L^2(\mathbb{R}_+; dm_\lambda)$:

$$\begin{aligned} \langle \Delta_\lambda f, f \rangle_{L^2(\mathbb{R}_+; dm_\lambda)} &\geq 0 \quad \forall f \in L^2(\mathbb{R}_+; dm_\lambda) \\ \langle \Delta_\lambda f, g \rangle_{L^2(\mathbb{R}_+; dm_\lambda)} &= \langle f, \Delta_\lambda g \rangle_{L^2(\mathbb{R}_+; dm_\lambda)} . \end{aligned}$$

Riesz Transforms associated to the Bessel Operator

- Akin to the Euclidean setting we define:

$$R_{\Delta_\lambda} f := \partial_x (\Delta_\lambda)^{-1/2} f$$

- One can show that the kernel of this operator is:

$$K(x,y) = -\frac{2\lambda}{\pi} \int_0^\pi \frac{(x - y\cos\theta)(\sin\theta)^{2\lambda-1}}{(x^2 + y^2 - 2xy\cos\theta)^{\lambda+1}}\, d\theta \quad x, y \in \mathbb{R}_+.$$

- This is a Calderón-Zygmund kernel on the space of homogenous type:
  i) for every $x, y \in \mathbb{R}_+$ with $x \neq y$,

$$|K(x,y)| \lesssim \frac{1}{m_\lambda(I(x, |x-y|))};$$

  ii) for every x, x_0, $y \in \mathbb{R}_+$ with $|x_0 - x| < |x_0 - y|/2$,

$$|K(y,x_0) - K(y,x)| + |K(x_0,y) - K(x,y)| \lesssim \frac{|x_0 - x|}{|x_0 - y|} \frac{1}{m_\lambda(I(x_0, |x_0 - y|))}.$$

BMO and the Hardy Space associated to the Bessel Operator

Definition (BMO Associated to the Bessel Operator)

A function $f \in L^1_{\text{loc}}(\mathbb{R}_+; dm_\lambda)$ belongs to $\text{BMO}(\mathbb{R}_+; dm_\lambda)$ if

$$\sup_{x,\, r \in \mathbb{R}_+} \frac{1}{m_\lambda(I(x,\, r))} \int_{I(x,\, r)} \left| f(y) - \frac{\int_{I(x,\, r)} f(z)\, dm_\lambda(z)}{m_\lambda(I(x, r))} \right| dm_\lambda(y) < \infty.$$

Definition (Hardy Space associated to the Bessel Operator)

$$H^1(\mathbb{R}_+; dm_\lambda) := \{f \in L^1(\mathbb{R}_+; dm_\lambda) : \ R_{\Delta_\lambda} f \in L^1(\mathbb{R}_+; dm_\lambda)\}$$

$$\|f\|_{H^1(\mathbb{R}_+; dm_\lambda)} := \|f\|_{L^1(\mathbb{R}_+; dm_\lambda)} + \|R_{\Delta_\lambda} f\|_{L^1(\mathbb{R}_+; dm_\lambda)}.$$

Theorem

The dual of $H^1(\mathbb{R}_+; dm_\lambda)$ is $BMO(\mathbb{R}_+; dm_\lambda)$.

BMO & Commutators

Let $[b, R_{\Delta_\lambda}]$ be the commutator defined by

$$[b, R_{\Delta_\lambda}]f(x) := b(x)R_{\Delta_\lambda}f(x) - R_{\Delta_\lambda}(bf)(x).$$

Theorem (Duong, Li, W., Yang, IUMJ, (2017))

Let $b \in \cup_{q>1} L^q_{\text{loc}}(\mathbb{R}_+; dm_\lambda)$ *and* $p \in (1, \infty)$.
(1) *If* $b \in \text{BMO}(\mathbb{R}_+; dm_\lambda)$, *then the commutator* $[b, R_{\Delta_\lambda}]$ *is bounded on* $L^p(\mathbb{R}_+; dm_\lambda)$ *with the operator norm*

$$\left\|[b, R_{\Delta_\lambda}]\right\|_{L^p(\mathbb{R}_+;dm_\lambda)\to L^p(\mathbb{R}_+;dm_\lambda)} \leq C\|b\|_{\text{BMO}(\mathbb{R}_+;dm_\lambda)}.$$

(2) *If* $[b, R_{\Delta_\lambda}]$ *is bounded on* $L^p(\mathbb{R}_+; dm_\lambda)$, *then* $b \in \text{BMO}(\mathbb{R}_+; dm_\lambda)$ *and*

$$\|b\|_{\text{BMO}(\mathbb{R}_+;dm_\lambda)} \leq C\left\|[b, R_{\Delta_\lambda}]\right\|_{L^p(\mathbb{R}_+;dm_\lambda)\to L^p(\mathbb{R}_+;dm_\lambda)}.$$

Hardy Spaces & Factorizations

Theorem (Duong, Li, W., Yang, IUMJ (2017))

Let $p \in (1, \infty)$ and p' be the conjugate of p. For any $f \in H^1(\mathbb{R}_+; dm_\lambda)$, there exist numbers $\{\alpha_j^k\}_{k,j}$, functions $\{g_j^k\}_{k,j} \subset L^p(\mathbb{R}_+; dm_\lambda)$ and $\{h_j^k\}_{k,j} \subset L^{p'}(\mathbb{R}_+; dm_\lambda)$ such that

$$f = \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \alpha_j^k \, \Pi(g_j^k, h_j^k)$$

in $H^1(\mathbb{R}_+; dm_\lambda)$, where the operator Π is: $\Pi(g, h) := g R_{\Delta_\lambda} h - h R^*_{\Delta_\lambda} g$. Moreover, there exists positive constants such that

$$\|f\|_{H^1(\mathbb{R}_+; dm_\lambda)} \approx \inf \left\{ \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} |\alpha_j^k| \left\| g_j^k \right\|_{L^p(\mathbb{R}_+; dm_\lambda)} \left\| h_j^k \right\|_{L^{p'}(\mathbb{R}_+; dm_\lambda)} \right\}.$$

Hilbert Transform Along a Parabola

The Hilbert transform along $\gamma(t) = (t, t^2)$ is defined as

$$H_\gamma(f)(x) := \text{p.v.} \int_{-\infty}^{\infty} f(x - \gamma(t)) \frac{dt}{t}, \quad x \in \mathbb{R}^2.$$

Definition

We call $Q \subset \mathbb{R}^2$ a parabolic cube if $Q = I_1 \times I_2$, where I_1 and I_2 are intervals on $\mathbb{R}$ and $|I_2| = |I_1|^2$.

Definition

Suppose $b \in L^1_{loc}(\mathbb{R}^2)$. b is in $\mathrm{BMO}_\gamma(\mathbb{R}^2)$ if

$$\|b\|_{\mathrm{BMO}_\gamma(\mathbb{R}^2)} := \sup_Q \frac{1}{|Q|} \int_Q |b(x) - b_Q| dx < \infty,$$

where the sup is taken over all parabolic cubes and $b_Q = \frac{1}{|Q|} \int_Q b(y) dy$.

Hilbert Transform Along a Parabola

Theorem (Bongers, Li, W. (2019))

Suppose $1 < p < \infty$. *There exists a positive constant* C_1 *such that for* $b \in \mathrm{BMO}_\gamma(\mathbb{R}^2)$, *we have*

$$\|[b, H_\gamma] : \ L^p(\mathbb{R}^2) \to L^p(\mathbb{R}^2)\| \leq C_1 \|b\|_{\mathrm{BMO}_\gamma(\mathbb{R}^2)}.$$

We do not know if the lower bound holds true. We can prove that if the commutator is bounded, then there is some necessary condition the symbol b must satisfy, but it isn't obvious that this new condition is the same as being in parabolic BMO.

Commutators and Lie Groups

Suppose $\mathcal{G}$ is a stratified nilpotent Lie group.
Recall that a connected, simply connected nilpotent Lie group $\mathcal{G}$ is said to be stratified if its left-invariant Lie algebra $\mathfrak{g}$ (assumed real and of finite dimension) admits a direct sum decomposition

$$\mathfrak{g} = \bigoplus_{i=1}^{k} V_i \text{ where } [V_1, V_i] = V_{i+1} \text{ for } i \leq k-1.$$

Let $\{X_j\}_{1\leq j\leq n}$ be a basis for the left-invariant vector fields of degree one on $\mathcal{G}$. Let $\Delta = \sum_{j=1}^{n} X_j^2$ be the sub-Laplacian on $\mathcal{G}$. Consider the j^{th} Riesz transform on $\mathcal{G}$ which is defined as $R_j := X_j(-\Delta)^{-\frac{1}{2}}$.

Commutators and Lie Groups

Definition

$$\mathrm{BMO}(\mathcal{G}) := \{b \in L^1_{loc}(\mathcal{G}) : \ \|b\|_{\mathrm{BMO}(\mathcal{G})} < \infty\},$$

where

$$\|b\|_{\mathrm{BMO}(\mathcal{G})} := \sup_B \frac{1}{|B|} \int_B |b(g) - b_B| dg.$$

and $b_B := \frac{1}{|B|} \int_B b(g)\, dg$, where B denotes the ball on $\mathcal{G}$ defined via a homogeneous norm ρ.

Theorem (Duong, Li, and W., J. Math. Pures Appl. (2019))

Suppose that $\mathcal{G}$ is a stratified nilpotent Lie group and that $1 < p < \infty$ and $j = 1, 2, \ldots, n$. Then the commutator of $b \in \mathrm{BMO}(\mathcal{G})$ and the Riesz transform R_j satisfies

$$\|[b, R_j] : L^p(\mathcal{G}) \to L^p(\mathcal{G})\| \approx \|b\|_{\mathrm{BMO}(\mathcal{G})}.$$

Commutators and Little BMO

We work in the multiparameter setting $\mathbb{R} \times \mathbb{R}$ where we study operators that are invariant under dilations in each variable separately.

Definition

A function $b \in L^1_{loc}(\mathbb{R}^2)$ is in $\mathrm{bmo}(\mathbb{R} \times \mathbb{R})$ if

$$\|b\|_{\mathrm{bmo}(\mathbb{R}\times\mathbb{R})} := \sup_{R \subset \mathbb{R}\times\mathbb{R}} \frac{1}{|R|} \iint_R |b(x_1, x_2) - b_R| dx_1 dx_2 < \infty,$$

where

$$b_R := \frac{1}{|R|} \iint_R b(x_1, x_2) dx_1 dx_2$$

is the mean value of b over the rectangle R.

It is well known that $\mathrm{bmo}(\mathbb{R} \times \mathbb{R})$ coincides with the space of integrable functions which are uniformly of bounded mean oscillation in each variable separately.

Commutators and Little BMO

We have the following equivalent characterizations for $\mathrm{bmo}(\mathbb{R} \times \mathbb{R})$.

Theorem (Ferguson–Sadosky)

Let $b \in L^1_{loc}(\mathbb{R}^2)$. The following conditions are equivalent:

(i) $b \in \mathrm{bmo}(\mathbb{R} \times \mathbb{R})$;

(ii) *The commutators $[b, H_1]$ and $[b, H_2]$ are both bounded on $L^2(\mathbb{R}^2)$;*

(iii) *The commutator $[b, H_1 H_2]$ is bounded on $L^2(\mathbb{R}^2)$.*

The proof of the above theorem is done via complex analysis techniques.

Atoms for Little $h^1(\mathbb{R}\times\mathbb{R})$

Definition (Ferguson–Sadosky)

An atom on $\mathbb{R}\times\mathbb{R}$ is a function $a\in L^\infty(\mathbb{R}^2)$ supported on a rectangle $R\subset\mathbb{R}\times\mathbb{R}$ with $\|a\|_\infty\leq |R|^{-1}$ and satisfying the cancellation property

$$\int_{\mathbb{R}^2} a(x_1,x_2)dx_1dx_2=0.$$

Let $Atom(\mathbb{R}\times\mathbb{R})$ denote the collection of all such atoms.

Definition

The atomic Hardy space $h^1(\mathbb{R}\times\mathbb{R})$ is defined as the set of functions of the form $f=\sum_i \alpha_i a_i$ with $\{a_i\}_i\subset Atom(\mathbb{R}\times\mathbb{R})$, $\{\alpha_i\}_i\subset\mathbb{C}$ and $\sum_i|\alpha_i|<\infty$. Moreover, $h^1(\mathbb{R}\times\mathbb{R})$ is equipped with the norm $\|f\|_{h^1(\mathbb{R}\times\mathbb{R})}:=\inf\sum_i|\alpha_i|$ where the infimum is taken over all possible decompositions of f.

Commutators and Little BMO

Theorem (Duong, Li, W. and D. Yang, Ann. Inst. Fourier (2018))

For every $f \in h^1(\mathbb{R}\times\mathbb{R})$, *there exist sequences* $\{\alpha_j^k\}_j \in \ell^1$ *and functions* $g_j^k, h_j^k \in L^2(\mathbb{R}^2)$ *such that*

$$f = \sum_{k=1}^{\infty}\sum_{j=1}^{\infty} \alpha_j^k \, \Pi\left(g_j^k, h_j^k\right)$$

in the sense of $h^1(\mathbb{R}\times\mathbb{R})$, *where* $\Pi(f,g)$ *is the bilinear form defined as*

$$\Pi(g,h) := hH_1H_2g - gH_1H_2h.$$

Moreover, we have that

$$\|f\|_{h^1(\mathbb{R}\times\mathbb{R})} \approx \inf\left\{\sum_{k=1}^{\infty}\sum_{j=1}^{\infty} \left|\alpha_j^k\right| \left\|g_j^k\right\|_{L^2} \left\|h_j^k\right\|_{L^2}\right\}.$$

The daydreams of cat herders

(Modified from the Original Dr. Fun Comic)

Thanks to the Organizers for Arranging the Meeting!

Thank You!