Remarks on Product VMO

or

Compactness of Little Hankel Operators

(joint with M. Lacey and E. Terwilleger)

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Nehari's Theorem

$$L^2(\mathbb{T}) = H^2(\mathbb{D}) \oplus H^2_-(\mathbb{D})$$

Let $\mathbb{P}_-$ denote the orthogonal projection of $L^2(\mathbb{T})$ onto $H^2_-(\mathbb{D})$, i.e.,

$$\mathbb{P}_- : L^2(\mathbb{T}) \to H^2_-(\mathbb{D}).$$

For $\varphi \in L^2(\mathbb{T})$, define the Hankel operator

$H_\varphi : H^2(\mathbb{D}) \longrightarrow H^2_-(\mathbb{D})$ by

$$H_\varphi f := \mathbb{P}_- M_\varphi f$$

with $M_\varphi f := \varphi f$.

Theorem. (Nehari, 1957) *H_φ is bounded iff there is a function $\psi \in L^\infty(\mathbb{T})$ for which*

$$\mathbb{P}_- \varphi = \mathbb{P}_- \psi.$$

In this case,

$$\begin{aligned}\|H_\varphi\| &= \inf\{\|\psi\|_\infty : \widehat{\varphi}(m) = \widehat{\psi}(m),\ m < 0\} \\ &= \|\mathbb{P}_- \varphi\|_{BMO(\mathbb{T})}.\end{aligned}$$

Hartman Compactness Criteria

Let $H^\infty(\mathbb{D})+C(\mathbb{T})$ denote the functions $\varphi \in L^\infty(\mathbb{T})$ such that $\varphi = f + g$, where $f \in H^\infty(\mathbb{D})$ and $g \in C(\mathbb{T})$.

Theorem. (Hartman, 1958) *Let $\varphi \in L^\infty$. TFAE:*

(i) H_φ is compact;

(ii) $\varphi \in H^\infty(\mathbb{D}) + C(\mathbb{T})$;

(iii) there exists $\psi \in C(\mathbb{T})$ such that $H_\varphi = H_\psi$.

This theorem, along with the fact

$$VMO(\mathbb{T}) = \{\xi + \tilde{\eta} : \xi, \eta \in C(\mathbb{T})\},$$

gives the following

Theorem. *Let $\varphi \in L^2(\mathbb{T})$. Then H_φ is compact iff $\mathbb{P}_-\varphi \in VMO(\mathbb{T})$.*

Little Hankel Operators on the Bi-Torus

$H^2_{\pm,\pm}(\mathbb{D}^2)$ is the space of square integrable functions which are (anti-) holomorphic in each variable separately.

Since

$$L^2(\mathbb{T}^2) = \oplus_{\varepsilon\in\{\pm,\pm\}} H^2_\varepsilon(\mathbb{D}^2).$$

define

$$\mathbb{P}_{\pm,\pm} : L^2(\mathbb{T}^2) \to H^2_{\pm,\pm}(\mathbb{D}^2).$$

The “little” Hankel operators from $H^2(\mathbb{D}^2)$ to $H^2_{-,-}(\mathbb{D}^2)$ are given by

$$h_\varphi := \mathbb{P}_{-,-} M_\varphi,$$

where $M_\varphi f := \varphi f$.

Main Results

Theorem 1 *The little Hankel operator h_φ is bounded iff there is a function $\psi \in L^\infty(\mathbb{T}^2)$ for which $\mathbb{P}_{-,-}\varphi = \mathbb{P}_{-,-}\psi$, and we have the equivalence*

$$\|h_\varphi\| \approx \inf\{\|\psi\|_\infty : \mathbb{P}_{-,-}\varphi = \mathbb{P}_{-,-}\psi\}$$
$$\approx \|\mathbb{P}_{-,-}\varphi\|_{BMO(\mathbb{T}^2)}.$$

Theorem 2 *The operator h_φ is compact iff $\mathbb{P}_{-,-}\varphi$ is in the closure of $C(\mathbb{T}^2)$ with respect to the BMO topology. This space we call $VMO(\mathbb{T}^2)$.*

Theorem 3 *Let $\varphi \in L^\infty(\mathbb{T}^2)$. Then the following are equivalent:*

(i) h_φ *is compact;*

(ii) $\varphi \in \mathcal{L}^\infty(\mathbb{T}^2) + C(\mathbb{T}^2)$;

(iii) $\exists$ *a* $g \in C(\mathbb{T}^2)$ *such that* $h_\varphi = h_g$.

Remarks

- Theorem 1 is just a restatement of the result by Ferguson and Lacey from 2002.

- The space $\mathcal{L}^\infty(\mathbb{T}^2)$ plays the role that $H^\infty(\mathbb{D})$ does in the one variable setting.

  This is the set of functions in $L^\infty(\mathbb{T}^2)$ that are in the kernel of $\mathbb{P}_{-,-}$.

  Can also be thought of as the set of functions that have an analytic extension in at least one variable.

Proof of Theorem 3

- This can be viewed as the extension of Hartman's Compactness Criterion to the bi-disk.

- The essential norm of an operator is given by

$$\|h_\varphi\|_e := \inf\{\|h_\varphi - K\|\},$$

with the infimum taken over all compact operators $K : H^2(\mathbb{D}^2) \to H_{-,-}(\mathbb{D}^2)$.

- Then $\|h_\varphi\|_e = 0$ iff h_φ is compact.

An even stronger statement is true:

Theorem 4 *Let $\varphi \in L^\infty(\mathbb{T}^2)$. Then*

$$\|h_\varphi\|_e \approx \mathrm{dist}_{L^\infty}(\varphi, \mathcal{L}^\infty(\mathbb{T}^2) + C(\mathbb{T}^2)).$$

The crucial fact is that $\mathcal{L}^\infty + C$ is a closed subspace.

Proof of Theorem 3 Continued

((ii)⇒(iii))

If $\varphi \in \mathcal{L}^\infty + C$, then $\varphi = \psi + g$ with $\psi \in \mathcal{L}^\infty(\mathbb{T}^2)$ and $g \in C(\mathbb{T}^2)$.

For any $f \in H^2(\mathbb{D}^2)$ we have

$$\begin{aligned} h_\varphi f &= \mathbb{P}_{-,-}\varphi f = \mathbb{P}_{-,-}[gf + \psi f] \\ &= \mathbb{P}_{-,-}gf + \mathbb{P}_{-,-}\psi f = h_g f, \end{aligned}$$

because $\psi f \in \mathcal{L}^2(\mathbb{T}^2)$ and $\mathbb{P}_{-,-}(\mathcal{L}^2(\mathbb{T}^2)) = 0$.

((iii)⇒(ii))

$\exists$ a function $g \in C(\mathbb{T}^2)$ such that $h_\varphi = h_g$. So

$$\mathbb{P}_{-,-}((\varphi - g)f) = 0 \qquad \forall f \in H^2(\mathbb{D}^2).$$

Taking $f = 1$ gives $\varphi - g \in \mathcal{L}^2(\mathbb{T}^2)$. By hypothesis, we have $\varphi - g \in L^\infty(\mathbb{T}^2)$, i.e. $\varphi \in \mathcal{L}^\infty(\mathbb{T}^2) + C(\mathbb{T}^2)$.

Proof of Theorem 3 Continued

((i)⇔(ii))

Follows immediately from Theorem 4.

h_φ is compact iff $\|h_\varphi\|_e = 0$.

But if $\|h_\varphi\|_e = 0$, by Theorem 4

$$\text{dist}_{L^\infty}(\varphi, \mathcal{L}^\infty + C) = 0.$$

So $\varphi \in \mathcal{L}^\infty + C$.

If $\varphi \in \mathcal{L}^\infty + C$, then

$$\text{dist}_{L^\infty}(\varphi, \mathcal{L}^\infty + C) = 0.$$

By Theorem 4 $\|h_\varphi\|_e = 0$, or h_φ is compact.

Proof of Theorem 4

- In the classic (one variable) setting Nehari's Theorem can be stated as

$$\|H_\varphi\| = \mathrm{dist}_{L^\infty}(\varphi, H^\infty(\mathbb{D})).$$

- We need something like this in the two variable setting.

  In two variables, Theorem 1 (Nehari's Theorem) can be restated as:

$$\|h_\varphi\| \approx \mathrm{dist}_{L^\infty}(\varphi, \mathcal{L}^\infty(\mathbb{T}^2))$$

  Proof of this fact follows immediately from Theorem 1 (work of Ferguson and Lacey).

We are also going to need a characterization of the space $\mathcal{L}^\infty + C$.

Similar to the one-variable case we have the following theorem.

Theorem 5 *$\mathcal{L}^\infty + C$ is a closed subspace of $L^\infty(\mathbb{T}^2)$, and moreover*

$$\mathcal{L}^\infty + C = \mathrm{clos}_{L^\infty}\left(\bigcup_{n,m=0}^{\infty} \overline{z}_1^n \overline{z}_2^m \mathcal{L}^\infty(\mathbb{T}^2)\right).$$

The above theorem will follow from

Theorem 6 *Let $C_{\mathcal{L}}(\mathbb{T}^2) := \mathcal{L}^\infty(\mathbb{T}^2) \cap C(\mathbb{T}^2)$ and $\varphi \in C(\mathbb{T}^2)$. Then*

$$\mathrm{dist}_{L^\infty}(\varphi, \mathcal{L}^\infty) = \mathrm{dist}_{L^\infty}(\varphi, C_{\mathcal{L}}).$$

Proof of Theorem 6

- $C_{\mathcal{L}}(\mathbb{T}^2) \subset \mathcal{L}^\infty(\mathbb{T}^2)$ implies

$$\text{dist}_{L^\infty}(\varphi, \mathcal{L}^\infty) \le \text{dist}_{L^\infty}(\varphi, C_{\mathcal{L}}).$$

- The other inequality follows from the same argument as in one variable.

  Let $\psi \in \mathcal{L}^\infty(\mathbb{T}^2)$.

  Using the harmonic extension of functions to the bi-disk we have

$$\begin{aligned} \|\varphi - \psi\|_\infty &\ge \lim_{r\to 1} \|(\varphi - \psi)_r\|_\infty \\ &\ge \lim_{r\to 1} (\|\varphi - \psi_r\|_\infty - \|\varphi - \varphi_r\|_\infty) \\ &= \lim_{r\to 1} \|\varphi - \psi_r\|_\infty. \end{aligned}$$

This gives

$$\text{dist}_{L^\infty}(\varphi, \mathcal{L}^\infty) \ge \text{dist}_{L^\infty}(\varphi, C_{\mathcal{L}}).$$

Proof of Theorem 5

- Theorem 6 gives that $C/C_{\mathcal{L}}$ has an isometric embedding in $L^\infty/\mathcal{L}^\infty$. So we may view it as a closed subspace of $L^\infty/\mathcal{L}^\infty$.

- Let $\rho : L^\infty \to L^\infty/\mathcal{L}^\infty$ be the natural quotient map. Then
$$\mathcal{L}^\infty + C = \rho^{-1}(C/C_{\mathcal{L}})$$
is closed.

- Finally,
$$\mathcal{L}^\infty + C = \text{clos}_{L^\infty}\left(\cup_{n,m=0}^{\infty}\overline{z}_1^n\overline{z}_2^m\mathcal{L}^\infty(\mathbb{T}^2)\right),$$
because continuous functions on $\mathbb{T}^2$ can be uniformly approximated by polynomials in z_1, z_2 and their conjugates.

- The proof is similar to the corresponding fact in the one variable case. Namely,
$$H^\infty + C = \text{clos}_{L^\infty}\left(\cup_{n=0}^{\infty}\overline{z}^n H^\infty\right).$$

Proof of Theorem 4

- Let $K : H^2(\mathbb{D}^2) \to H^2_{-,-}(\mathbb{D}^2)$ be a compact operator.

  We want to estimate $\|h_\varphi - K\|$ from below.

- Let S_j be multiplication by the variable z_j. Note S_j is a contraction. Then

$$
\begin{aligned}
\|h_\varphi - K\| &\geq \|(h_\varphi - K)S_1^n S_2^m\| \\
&\geq \|h_\varphi S_1^n S_2^m\| - \|K S_1^n S_2^m\| \\
&= \|h_{z_1^n z_2^m \varphi}\| - \|K S_1^n S_2^m\| \\
&\gtrsim \text{dist}_{L^\infty}(\varphi, \overline{z}_1^n \overline{z}_2^m \mathcal{L}^\infty) - \|K S_1^n S_2^m\| \\
&\geq \text{dist}_{L^\infty}(\varphi, \mathcal{L}^\infty + C) - \|K S_1^n S_2^m\|.
\end{aligned}
$$

- Used the fact that $\|h_\varphi\| \approx \text{dist}_{L^\infty}(\varphi, \mathcal{L}^\infty)$.

- Also used that $\mathcal{L}^\infty + C$ is a closed subspace.

Proof of Theorem 4 Continued

We need one Lemma

Lemma 7 *Let K be a compact operator from $H^2(\mathbb{D}^2)$ to $H^2_{-,-}(\mathbb{D}^2)$. Then*

$$\|KS_1^n S_2^m\| \to 0$$

as $n, m \to \infty$.

- Implies $\|h_\varphi - K\| \gtrsim \text{dist}_{L^\infty}(\varphi, \mathcal{L}^\infty + C)$ for any compact operator K. Which gives

$$\|h_\varphi\|_e \gtrsim \text{dist}_{L^\infty}(\varphi, \mathcal{L}^\infty + C).$$

Now we need to estimate $\|h_\varphi - K\|$ from above.

- Let g be a trigonometric polynomial. Then h_g is a compact (finite rank) operator. So

$$\|h_\varphi\|_e \leq \inf_{g \in C} \|h_\varphi - h_g\| = \inf_{g \in C} \|h_{\varphi - g}\|.$$

Proof of Theorem 4 Continued

- By restating Nehari's Theorem we have

$$\begin{aligned}\|h_\varphi\|_e &\lesssim \inf_{g \in C, \psi \in \mathcal{L}^\infty} \|\varphi - g - \psi\|_\infty \\ &\lesssim \mathrm{dist}_{L^\infty}(\varphi, \mathcal{L}^\infty + C).\end{aligned}$$