Bounded Analytic Projections and the Corona Problem

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Formulation of the Problem

- Let E and E_* be separable complex Hilbert spaces.
- Let Ω be a domain in $\mathbb{C}^n$.
- $H^2_E(\Omega)$ is the analytic Hardy space with values in the Hilbert space E.
- $H^\infty_{E_*\to E}(\Omega)$ is the collection of all bounded operator-valued functions.

$$F(z) : E_* \to E \text{ and } \|F\|_{H^\infty_{E_*\to E}(\Omega)} := \sup_{z\in\Omega} \|F(z)\|_{E_*\to E}$$

Question (Operator Corona Problem)

Let $F \in H^\infty_{E_*\to E}(\Omega)$. Can we find, preferably local, necessary and sufficient conditions on F so that it has an analytic left inverse? Namely, what conditions imply the existence of a function $G \in H^\infty_{E\to E_*}(\Omega)$ such that

$$G(z)F(z) \equiv I \quad \forall z \in \Omega.$$

A simple necessary condition is:

$$F^*(z)F(z) \geq \delta^2 I \quad \forall z \in \Omega.$$

Connection to the Usual Corona Problem

Let $\Omega = \mathbb{D}$, the unit disc in the complex plane. The Operator Corona Problem is a more general question based on the following:

Question (Corona Problem)

Suppose that $f_1, \ldots, f_n \in H^\infty(\mathbb{D})$ *with* $\|f_j\|_\infty \leq 1$ *and*

$$\sum_{j=1}^{n} |f_j(z)|^2 \geq \delta^2 \quad \forall z \in \mathbb{D}.$$

Do there exist $g_j \in H^\infty(\mathbb{D})$ *such that*

$$\sum_{j=1}^{n} f_j(z) g_j(z) \equiv 1 \quad \forall z \in \mathbb{D}?$$

Take $F(z) = (f_1(z), \ldots, f_n(z))^T$ in the Operator Corona Problem to recover this question.

Known Results

Let $\Omega = \mathbb{D}$, the unit disc in the complex plane.

- When $E_* = \mathbb{C}$ and $\dim E < \infty$.
  - In 1962 Carleson demonstrated that the simple necessary condition is sufficient.
  - In 1979 Wolff gave a simpler compact proof of Carleson's result.
- When $E_* = \mathbb{C}, \dim E = \infty$.
  - Rosenblum, Tolokonnikov, and Uchiyama independently gave proofs.
- When $\dim E_* < \infty$ and $\dim E = \infty$. (Matrix Corona Problem)
  - Fuhrmann and Vasyunin independently demonstrated this.
- When $\dim E = \dim E_* = \infty$. (Operator Corona Problem)
  - In 1988 Treil constructed a counter example which indicates that the necessary condition is no longer sufficient.
  - In 2004 he gave another construction which demonstrated the same phenomenon.

When $n \geq 2$ and $\Omega \subset \mathbb{C}^n$ (e.g. $\mathbb{B}$ or $\mathbb{D}^n$) the H^∞ Corona Problem is open.

Nikolski's Lemma

Lemma (Nikolski's Lemma)

Let $F \in H^\infty_{E_* \to E}(\Omega)$ *satisfy*

$$F^*(z)F(z) \geq \delta^2 I, \quad \forall z \in \Omega.$$

Then F *is left invertible in* $H^\infty_{E_* \to E}(\Omega)$ *(i.e., there exists* $G \in H^\infty_{E \to E_*}(\Omega)$ *such that* $GF \equiv I$*) if and only if there exists a function* $\mathcal{P} \in H^\infty_{E \to E}(\Omega)$ *whose values are projections (not necessarily orthogonal) onto* $F(z)E$ *for all* $z \in \Omega$*.*

Key Point: Finding a left inverse is replaced with constructing a bounded analytic projection-valued function that takes a prescribed range.

Proof of Nikolski's Lemma

"$\Rightarrow$"

- Let F be left invertible in $H^\infty_{E_* \to E}(\Omega)$, and let G any left inverse.
- Define $\mathcal{P} \in H^\infty_{E \to E}(\Omega)$ by

$$\mathcal{P}(z) = F(z)G(z).$$

- Note that

$$\mathcal{P}^2(z) = F(z)G(z)F(z)G(z) = F(z)IG(z) = F(z)G(z) = \mathcal{P}(z).$$

- The values of $\mathcal{P}$ are projections.
- Since $GF \equiv I$

$$G(z)E = E_* \quad \forall z \in \Omega.$$

- Therefore

$$\mathcal{P}(z)E = F(z)G(z)E = F(z)E_* \quad \forall z \in \Omega$$

- $\mathcal{P}(z)$ is a bounded analytic projection onto $F(z)E_*$.

Proof of Nikolski's Lemma

"$\Leftarrow$"

- Suppose there exists a projection-valued function $\mathcal{P} \in H^\infty_{E \to E}$, whose values are projections onto $F(z)E_*$ for all $z \in \Omega$.
- In a neighborhood of each point $z_0 \in \Omega$ the function $F(z)$ has an analytic left inverse.
  - Let $G_0 : E \longrightarrow E_*$ be a constant left inverse to the operator $F(z_0)$, i.e., $G_0 F(z_0) = I$.
  - Then
  $$G_0 F(z) = I - G_0(F(z_0) - F(z)).$$
  - The inverse of $G_0 F(z)$ is given by the analytic function
  $$A(z) := \sum_{k=0}^{\infty} [G_0 \cdot (F(z_0) - F(z))]^k$$
  defined in a neighborhood of z_0.
- $A(z)G_0$ is a local analytic left inverse of $F(z)$.

Proof of Nikolski's Lemma

- For a fixed $z \in \Omega$ the operator $F(z)$ is left invertible.
- It is invertible if we treat it as an operator from E_* to $F(z)E_*$.
- Let $F^\dagger(z) : F(z)E_* \to E_*$ be the inverse of the "restricted" $F(z)$.
- For any (not necessarily analytic) left inverse $\widetilde{G}(z)$ of $F(z)$

$$\widetilde{G}(z) \big|_{F(z)E_*} = F^\dagger(z) \big|_{F(z)E_*} .$$

- Since $\mathcal{P}(z)$ is a projection onto $F(z)E_*$, the function G,

$$G(z) := F^\dagger(z)\mathcal{P}(z)$$

  is well defined and bounded since both $F^\dagger$ and $\mathcal{P}$ are bounded.
- We have $G(z)F(z) \equiv I$. It only remains to show that G is analytic.
  - Fix $z_0 \in \Omega$ and let $G_{z_0}(z)$ be a *local* analytic left inverse of $F(z)$ defined in a neighborhood of z_0.
  - Then

$$G(z) = F^\dagger(z)\mathcal{P}(z) = G_{z_0}(z)$$

    in a neighborhood of z_0. So $G(z)$ is analytic there.
- Since z_0 is arbitrary, G is analytic in Ω.

The Corona Condition and Projections in $\mathbb{D}$

Let $F \in H^\infty_{E_* \to E}(\mathbb{D})$ be such that

$$F^*(z)F(z) \geq \delta^2 I \quad \forall z \in \mathbb{D}.$$

Set

$$\Pi(z) := F(z)(F^*(z)F(z))^{-1}F^*(z) \quad \forall z \in \mathbb{D}.$$

Note that

$$\begin{aligned}
\Pi(z) &= \Pi(z)^* \\
\Pi^2(z) &= F(z)(F^*(z)F(z))^{-1}F^*(z)F(z)(F^*(z)F(z))^{-1}F^*(z) \\
&= F(z)(F^*(z)F(z))^{-1}F^*(z) = \Pi(z).
\end{aligned}$$

The values of Π are orthogonal projections with $\operatorname{Ran}\Pi(z) = \operatorname{Ran} F(z)$. They are **not analytic**. Direct computation demonstrates that

$$\Pi(z)\partial\Pi(z) = 0 \quad \forall z \in \mathbb{D}.$$

Main Theorem

Theorem (S. Treil, BW)

Let $F \in H^\infty_{E_* \to E}(\mathbb{D})$ satisfy the Corona Condition $F^*F \geq \delta^2 I$. Assume that there exists a bounded non-negative subharmonic function φ such that

$$\Delta\varphi(z) \geq \|\partial\Pi(z)\|^2 \quad z \in \mathbb{D}.$$

Then F has a holomorphic left inverse $G \in H^\infty_{E \to E_*}(\mathbb{D})$.
Moreover, if the function φ satisfies

$$0 \leq \varphi(z) \leq K \quad \forall z \in \mathbb{D},$$

then one can find the left inverse G satisfying

$$\|G\|_\infty \leq \delta^{-1}\left(1 + 2\sqrt{(Ke^{K+1}+1)Ke^{K+1}}\right).$$

The Corona Condition and Projections

Method of Proof:

- Construct a bounded analytic projection $\mathcal{P}(z)$ with
$$\operatorname{Ran}\mathcal{P}(z) = \operatorname{Ran} F(z) \quad \forall z \in \mathbb{D}.$$
- Apply Nikolski's Lemma to see that F is left invertible.
- Use the projection $\Pi(z)$ as an initial guess for $\mathcal{P}(z)$.
- Key Idea: Find some bounded operator-valued function $V(z) : E \to E$ that we can use to "correct" the initial guess of $\Pi(z)$ to be holomorphic. Set $\mathcal{P}(z) = \Pi(z) - \Pi(z)V(z)(I - \Pi(z))$.

Lemma

Let Π be an orthogonal projection in a Hilbert space H. Then any projection $\mathcal{P}$ onto $\operatorname{Ran}\Pi$ can be represented as

$$\mathcal{P} = \Pi + \Pi V(I - \Pi),$$

where $V \in B(H)$.

Finding V: Reduction to a Bilinear Form

- We want $\Pi - \Pi V(I-\Pi) \in H^\infty_{E\to E}(\mathbb{D})$.
- Follows from the equality:

$$\int_{\mathbb{T}} \langle \Pi h_1, h_2\rangle dm = \int_{\mathbb{T}} \langle \Pi V(I-\Pi)h_1, h_2\rangle dm$$

to hold for all $h_1 \in H^2_E(\mathbb{D})$ and $h_2 \in H^2_E(\mathbb{D})^\perp$.

- Apply Green's formula to the left hand side:

$$\begin{aligned} \int_{\mathbb{T}} \langle \Pi h_1, h_2\rangle dm &= \frac{4}{2\pi}\int_{\mathbb{D}} \partial\overline{\partial}\langle \Pi h_1, h_2\rangle \log\left(\frac{1}{|z|}\right) dxdy \\ &= \int_{\mathbb{D}} \partial\langle \overline{\partial}\Pi h_1, h_2\rangle d\mu(z). \end{aligned}$$

Here we used the harmonic extensions of h_1 and h_2 with h_2 being anti-analytic and $h_2(0)=0$.

Finding V: Reduction to a Bilinear Form

- $\Pi\partial\Pi = 0$ implies $\Pi(\overline{\partial}\Pi)(I - \Pi) = \overline{\partial}\Pi$.
- Define $\xi_1 := (I - \Pi)h_1$ and $\xi_2 := \Pi h_2$. Then

$$\int_{\mathbb{D}} \partial\langle\overline{\partial}\Pi h_1, h_2\rangle d\mu(z) = \int_{\mathbb{D}} \partial\langle\overline{\partial}\Pi\xi_1, \xi_2\rangle d\mu(z) := L(\xi_1, \xi_2).$$

- The bilinear form L is a Hankel form, i.e., $L(z\xi_1, \xi_2) = L(\xi_1, \bar{z}\xi_2)$.
- Suppose that we are able to prove the estimate

$$|L(\xi_1, \xi_2)| \leq C\|\xi_1\|_2\|\xi_2\|_2.$$

We then can find V by applying an appropriate version of Nehari's Theorem to the Hankel form L.

Finding V: Reduction to a Bilinear Form

- There exists an operator-valued function $V \in L^{\infty}_{E\to E}(\mathbb{T})$ such that

$$L(\xi_1, \xi_2) = \int_{\mathbb{T}} \langle V\xi_1, \xi_2 \rangle dm.$$

- Recalling the definition of L and ξ_1, ξ_2 we get

$$\int_{\mathbb{T}} \langle \Pi V(I - \Pi)h_1, h_2 \rangle dm = L(\xi_1, \xi_2) = \int_{\mathbb{T}} \langle \Pi h_1, h_2 \rangle dm$$

  for all $h_1 \in H^2_E(\mathbb{D})$ and all $h_2 \in H^2_E(\mathbb{D})^{\perp}$.
- Gives $\mathcal{P}(z) := \Pi(z) - \Pi(z)V(z)(I - \Pi(z)) \in H^{\infty}_{E\to E}(\mathbb{D})$.

Main Point: We only need to prove the estimate

$$|L(\xi_1, \xi_2)| \leq C\|\xi_1\|_2\|\xi_2\|_2.$$

Estimating the Bilinear Form

- The Wolff approach to the Corona Problem depends upon demonstrating certain Embedding theorems. We will use a similar idea to show $|L(\xi_1,\xi_2)| \leq C\|\xi_1\|_2\|\xi_2\|_2$.
- First observe that

$$\begin{aligned} L(\xi_1,\xi_2) &= \int_{\mathbb{D}} \partial\langle\overline{\partial}\Pi\xi_1,\xi_2\rangle d\mu(z) = \int_{\mathbb{D}} \langle\partial\overline{\partial}\Pi\xi_1,\xi_2\rangle d\mu \\ &+ \int_{\mathbb{D}} \langle\overline{\partial}\Pi\partial\xi_1,\xi_2\rangle d\mu + \int_{\mathbb{D}} \langle\overline{\partial}\Pi\xi_1,\overline{\partial}\xi_2\rangle d\mu \\ &:= \mathrm{I} + \mathrm{II} + \mathrm{III}. \end{aligned}$$

- $\Pi\partial\Pi = 0$ implies that $\mathrm{I} \equiv 0$.
- II and III are symmetric. Only need to estimate one of them.

The Embedding Theorem

Lemma (Embedding Theorem for Holomorphic Vector Bundles)

Let φ be a non-negative bounded subharmonic function in $\mathbb{D}$ satisfying

$$\Delta\varphi(z) \geq \|\partial\Pi(z)\|^2, \qquad \forall z \in \mathbb{D},$$

and let $K = \|\varphi\|_\infty$. Then for all ξ_1 of the form $\xi_1 = (I - \Pi)h$, $h \in H^2_E(\mathbb{D})$ we have

$$\int_{\mathbb{D}} \Delta\varphi \, \|\xi_1\|^2 \, d\mu \leq eKe^K \|\xi_1\|_2^2$$

and

$$\int_{\mathbb{D}} \|\partial\xi_1\|^2 \, d\mu \leq (1 + eKe^K)\|\xi_1\|_2^2.$$

We have a similar estimate for ξ_2. Only replace ∂ by $\overline{\partial}$.

Estimating the Second Integral

$$
\begin{aligned}
|II| &= \left| \int_{\mathbb{D}} \langle \overline{\partial} \Pi \partial \xi_1, \xi_2 \rangle \, d\mu \right| \\
&\leq \int_{\mathbb{D}} \left| \langle \overline{\partial} \Pi \xi_1, \xi_2 \rangle \right| d\mu \\
&\leq \left(\int_{\mathbb{D}} \|\overline{\partial} \Pi\|^2 \|\xi_2\|^2 d\mu \right)^{1/2} \left(\int_{\mathbb{D}} \|\partial \xi_1\|^2 d\mu \right)^{1/2}
\end{aligned}
$$

Here we used Cauchy-Schwarz applied to vectors and integrals. Using that $\|\overline{\partial}\Pi\|^2 \leq \Delta\varphi$ and the Embedding Lemma gives

$$
|II| = \left(eKe^K \|\xi_2\|_2^2 \right)^{1/2} \left((1 + eKe^K) \|\xi_1\|_2^2 \right)^{1/2}.
$$

This proves $|L(\xi_1, \xi_2)| \leq C \|\xi_1\|_2 \|\xi_2\|_2$.

Existence of φ

- The Main Theorem and the Embedding Lemma were dependent upon the existence of a function φ that satisfied:

$$\Delta\varphi(z) \geq \|\partial\Pi(z)\|^2 \quad \forall z \in \mathbb{D}.$$

- The condition on φ simply means the Green potential

$$\mathcal{G}(\lambda) := \frac{2}{\pi} \iint_{\mathbb{D}} \ln\left|\frac{z-\lambda}{1-\overline{\lambda}z}\right| \|\partial\Pi(z)\|^2 dxdy$$

  is uniformly bounded in the disk $\mathbb{D}$.
- The are several possible candidates for such a function:
  - Direct computation shows that $\varphi(z) = C\,\mathrm{Tr}(F^*(z)F(z))$ works. Doesn't give good estimates in terms of the constants.
  - The function $\varphi(z) = \ln\det(F^*(z)F(z))$ also works and gives better estimates.