Homework 1

1. Prove that for any complex numbers z and w that

$$\begin{aligned} |z+w|^2 &= |z|^2+|w|^2+2\,\mathrm{Re}(z\overline{w}) \\ |z+w|^2+|z-w|^2 &= 2\,|z|^2+2\,|w|^2 . \end{aligned}$$

2. Prove that

$$1-\left|\frac{z-w}{1-z\overline{w}}\right|^2=\frac{(1-|z|^2)(1-|w|^2)}{|1-\overline{z}w|^2}.$$

3. Show that it is impossible to find a total ordering on $\mathbb{C}$. In other words, one can not find a relation $>$ between complex numbers so that

- For any two complex numbers z and w, one and only one of the following is true: $z>w$, $w>z$ and $z=w$.
- For all $z_1,z_2,z_3\in\mathbb{C}$, the relation $z_1>z_2$ implies $z_1+z_3>z_2+z_3$.
- For all $z_1z_2,z_3\in\mathbb{C}$ with $z_3>0$ then $z_1>z_2$ implies $z_1z_3>z_2z_3$.

4. Prove Lagrange's identity:

$$\left|\sum_{j=1}^n z_jw_j\right|^2=\sum_{j=1}^n|z_j|^2\sum_{j=1}^n|w_j|^2-\sum_{1\le j<k\le n}|z_j\overline{w}_k-\overline{w}_jz_k|^2 .$$

Deduce the Cauchy-Schwarz inequality from this identity.

5. Prove that the function $z\to|z|$ is continuous.

6. Let z,w be two complex numbers such that $\overline{z}w\neq 1$. Prove that:

$$\begin{aligned} \left|\frac{z-w}{1-\overline{z}w}\right|&<1 \quad \text{if} \quad |z|<1 \text{ and } |w|<1 \\ \left|\frac{z-w}{1-\overline{z}w}\right|&=1 \quad \text{if} \quad |z|=1 \text{ or } |w|=1 \end{aligned}$$

7. Show in polar coordinate that the Cauchy-Riemann equations take the form:

$$\begin{aligned} \frac{\partial u}{\partial r} &= \frac{1}{r}\frac{\partial v}{\partial\theta} \\ \frac{1}{r}\frac{\partial u}{\partial\theta} &= -\frac{\partial v}{\partial r} \end{aligned}$$

8. Prove the following version of the chain rule assuming everything is defined:

$$\begin{aligned} \frac{\partial}{\partial z}(f\circ g) &= \frac{\partial f}{\partial w}\frac{\partial g}{\partial z}+\frac{\partial f}{\partial\overline{w}}\frac{\partial\overline{g}}{\partial z} \\ \frac{\partial}{\partial\overline{z}}(f\circ g) &= \frac{\partial f}{\partial w}\frac{\partial g}{\partial\overline{z}}+\frac{\partial f}{\partial\overline{w}}\frac{\partial\overline{g}}{\partial\overline{z}} \end{aligned}$$

9. Prove that if f is holomorphic on $U \subset \mathbb{C}$ then:

$$\begin{aligned} \Delta(|f|^2) &= 4\left|\frac{\partial f}{\partial z}\right|^2 \\ \Delta(|f|^p) &= p^2\,|f|^{p-2}\left|\frac{\partial f}{\partial z}\right|^2 \text{ provided } f \text{ is nonvanishing.} \end{aligned}$$

10. If f is C^1 then:

$$\overline{\frac{\partial f}{\partial z}} = \frac{\partial \overline{f}}{\partial \overline{z}}.$$

11. Let $f : \mathbb{C} \to \mathbb{C}$ be a polynomial. Suppose that $\frac{\partial f}{\partial z} = 0$ and $\frac{\partial f}{\partial \overline{z}} = 0$, prove that f is constant.

12. Suppose that f is holomorphic in a region U. Prove that in any of the following cases:

- $\operatorname{Re} f$ is constant;
- $\operatorname{Im} f$ is constant;
- $|f|$ is constant;

then f is contant.