Homework 2

1. Let γ be a smooth curve in $\mathbb{C}$ parameterized by $z(t) : [a, b] \to \mathbb{C}$. Let γ^- denote the curve with the same image as γ but with the reverse orientation. Prove that for any continuous function f on γ that:

$$\int_\gamma f(z)dz = -\int_{\gamma^-} f(z)dz.$$

2. Evaluate the integrals

$$\int_\gamma z^n dz$$

for all integers n and γ any circle centered at the origin with the counter clockwise orientation.

3. Show that if $|a| < r < |b|$ then:

$$\int_\gamma \frac{1}{(z-a)(z-b)}dz = \frac{2\pi i}{a-b}$$

where γ denotes the circle centered at the origin of radius r with the positive orientation.

4. Compute the following complex line integrals:

(a) $\int_\gamma (\overline{z} + z^2\overline{z})dz$ where γ is the square centered at the origin with side length R with clockwise orientation;

(b) $\int_\gamma \frac{z}{8+z^2}dz$ where γ is the triangle with vertices $1, i, -i$ and γ is equipped with counter-clockwise orientation.

5. Let f be holomorphic in an open set U which is the interior of a disc or a rectangle. Let $\gamma : [0, 1] \to U$ be a C^1 curve satisfying $\gamma(0) = \gamma(1)$. Prove that:

$$\int_\gamma f(z)dz = 0.$$

6. Let f be a holomorphic function on an open set U which is the interior of a disc or a rectangle. Let $p, q \in U$. Let $\gamma_j : [a, b] \to U$, $j = 1, 2$ be C^1 curves such that $\gamma_j(a) = p$ and $\gamma_j(b) = q$. Show that

$$\int_{\gamma_1} f(z)dz = \int_{\gamma_2} f(z)dz.$$

7. Prove that if $U \subset \mathbb{C}$ and if $f : U \to \mathbb{C}$ has a complex derivative at each point of U, then f is continuous at each point of U.

8. A function $\gamma : [a, b] \to \mathbb{C}$ for $[a, b] \subset \mathbb{R}$ is of bounded variation if there is a constant M such that for any partition $P = \{a = t_0 < t_1 < \cdots < t_m = b\}$ of $[a, b]$:

$$v(\gamma, P) = \sum_{k=1}^{m} |\gamma(t_k) - \gamma(t_{k-1})| \leq M.$$

The total variation of γ, $V(\gamma)$ is defined by $V(\gamma) = \sup\{v(\gamma, P)$:P is a partition of $[a, b]\}$. Prove the following statements:

- γ is of bounded variation if and only if $\operatorname{Re}\gamma$ and $\operatorname{Im}\gamma$ are of bounded variation.

- If γ is real-valued and non-decreasing, then γ is of bounded variation and $V(\gamma) = \gamma(b) - \gamma(a)$;
- If $\gamma : [a, b] \to \mathbb{C}$ is piecewise smooth then γ is of bounded variation and:

$$V(\gamma) = \int_a^b |\gamma'(t)|\, dt.$$

9. Prove the following integration by parts formula: Let f and g be analytic in U and let γ be a C^1 curve with $\gamma : [a, b] \to \mathbb{C}$. Then:

$$\int_\gamma fg'\, dz = f(b)g(b) - f(a)g(a) - \int_\gamma f'g\, dz.$$

10. Let $U \subset \mathbb{C}$ be an open disc with center 0. Let f be holomoprhic on U. If $z \in U$, then define $\gamma_z(t) = tz$ for $0 \leq t \leq 1$. Define $F(z) = \int_{\gamma_z} f(w)dw$. Prove that F is a primitive of f.